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ERRATUM: FREE DIVISORS IN A PENCIL OF CURVES

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In the paper "Free divisors in a pencil of curves", I wrongly said that the Jacobian ideal $J_{\nabla D_k} \subset S$ generated by the partial derivatives of D_k is locally a complete intersection. This is not always true, as can be seen for instance in [2] or in [1, section 1.3].

Because of this error, remark 2.4, Theorem 2.7 and Theorem 2.8 are not true as they were formulated. In remark 2.4, the phrase "the local ideals $(\nabla f \wedge \nabla g)_p$ and $(\nabla f)_p$ coincide" is true when the Jacobian ideal of f is locally a complete intersection at p (for instance when $f = 0$ is a union of lines), but not in general.

Actually, the condition that the Jacobian ideal of a reduced plane curve C is a local complete intersection is equivalent to the claim that any singularity of C is weighted homogeneous (see again [1, section 1.3]).

This hypothesis concerning the nature of the singularities of the curves in the pencil must be added in order to correct theorems 2.7 and 2.8.

The set of all the singularities of all the singular members of the pencil $\mathcal{C}(f,g)$ is denoted by $\mathrm{Sing}(\mathcal{C})$.

A correct statement for Theorem 2.7 is the following one:

Theorem 2.7 *Assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth, $n \geq 1$ and $k > 1$. Then, D_k is free with exponents $(2n-2, n(k-2)+1)$ if and only if $D_k \supseteq D^{\mathrm{sg}}$ and $J_{\nabla D_k}$ is locally a complete intersection at every $p \in \mathrm{Sing}(\mathcal{C})$.*

Proof. Let us remark first that D_k is free with exponents $(2n-2, n(k-2)+1)$ if and only if the zero set Z_k of the "canonical section" $s_{\delta,k}$ is empty. Indeed, if $Z_k = \emptyset$, then $\mathrm{H}^1(\mathcal{T}_{D_k}(m)) = 0$ for all $m \in \mathbb{Z}$ and, by Horrocks' criterion, this implies that D_k is free with exponents $(2n-2, n(k-2)+1)$. The other direction is straightforward.

According to lemma 2.6, $Z_k = \emptyset$ if and only if $c_2(\mathcal{J}_{\nabla D_k}) = n^2(k-1)^2 + 3(n-1)^2$. Moreover it is well-known (see [1, section 1.3] for instance) that the length of the Jacobian scheme of D_k is $c_2(\mathcal{J}_{\nabla D_k}) = \sum_{p \in \mathrm{Sing}(D_k)} \tau_p(D_k)$, where $\tau_p(D_k)$ is the Tjurina number of D_k at $p \in D_k$ (this number $\tau_p(D_k)$ is the length of the subscheme of the Jacobian scheme supported by p). Then, to prove the theorem, we show below that $\sum_{p \in \mathrm{Sing}(D_k)} \tau_p(D_k) = n^2(k-1)^2 + 3(n-1)^2$ if and only if $D_k \supseteq D^{\mathrm{sg}}$ and $J_{\nabla D_k}$ is locally a complete intersection at every $p \in \mathrm{Sing}(\mathcal{C})$.

The Jacobian scheme of D_k is supported by the base locus B of the pencil and by the singular points of the k curves forming D_k. The syzygy $\nabla f \wedge \nabla g$ of $J_{\nabla D_k}$ does not vanish at $p \in B$; this implies that $J_{\nabla D_k}$ is locally a complete intersection at $p \in B$; according to [1, section 1.3], this gives $\tau_p(D_k) = \mu_p(D_k)$, where this last number is the Milnor number of D_k at p. Since p is an ordinary singular point of multiplicity k, we obtain $\mu_p(D_k) = (k-1)^2$. Then $\sum_{p \in B} \tau_p(D_k) = n^2(k-1)^2$.

Let us compute now $\sum_{p \in \mathrm{Sing}(D_k) \setminus B} \tau_p(D_k)$.

Let $C_p \subset D_k$ be the unique curve in the pencil singular at $p \in \mathrm{Sing}(D_k) \setminus B$. We can verify without difficulties that their Jacobian ideals coincide locally at $p \in \mathrm{Sing}(D_k) \setminus B$; in particular $\tau_p(D_k) = \tau_p(C_p)$ and $\sum_{p \in \mathrm{Sing}(D_k) \setminus B} \tau_p(D_k) = \sum_{p \in \mathrm{Sing}(D_k) \setminus B} \tau_p(C_p)$.

Let $I = (\nabla f \wedge \nabla g)$ be the ideal generated by the two by two minors of the 3×2 matrix $(\nabla f, \nabla g)$ defining the scheme $\mathrm{sg}(\mathcal{F})$. Let $\mathrm{sg}(\mathcal{F})_p$ be the subscheme of $\mathrm{sg}(\mathcal{F})$ supported by the point p. We have seen in lemma 2.2 that $\mathrm{sg}(\mathcal{F})$ is supported by the whole set of singular points of the pencil and that $l(\mathrm{sg}(\mathcal{F})) = \sum_{p\in \mathrm{Sing}(\mathcal{C})} l(\mathrm{sg}(\mathcal{F})_p) = 3(n-1)^2$.

Let us consider the situation in a fixed point $p \in \mathrm{Sing}(D_k) \setminus B$. To simplify the notation, assume that $f = 0$ is an equation for C_p. Then the other curves of the pencil do not pass through p; in particular, $g(p) \neq 0$. Since $< \nabla f \wedge \nabla g, \nabla g >= 0$, ∇g is a syzygy of I that does not vanish at $p \in \mathrm{Sing}(D_k) \setminus B$. This implies that I is locally a complete intersection at p. Since the ideal I_p is obtained by taking the two by two minors of the matrix $(\nabla f, \nabla g)$ in the local ring S_p, the inclusion $I_p \subset J_{\nabla f,p}$ is straightforward; this inclusion implies $\tau_p(C_p) \leq l(\mathrm{sg}(\mathcal{F})_p)$ because $l(\mathrm{sg}(\mathcal{F})_p) = l(S_p/I_p)$.

Then $Z_k = \emptyset$ if and only if $\sum_{p\in \mathrm{Sing}(D_k)\setminus B} \tau_p(C_p) = \sum_{p\in \mathrm{Sing}(\mathcal{C})} l(\mathrm{sg}(\mathcal{F})_p)$. And this equality is verified if and only if $\mathrm{Sing}(D_k) \setminus B = \mathrm{Sing}(\mathcal{C})$ and $l(\mathrm{sg}(\mathcal{F})_p) = \tau_p(C_p)$ for all $p \in \mathrm{Sing}(\mathcal{C})$. The second equality is equivalent to the equality $I_p = J_{\nabla f,p}$, which implies that the Jacobian ideal of C_p is locally a complete intersection at p.

Since the Jacobian ideals of D_k and C_p coincide locally at $p \in \mathrm{Sing}(D_k) \setminus B$, this proves that $Z_k = \emptyset$ if and only if D_k contains all the singular members of the pencil ($\mathrm{Sing}(D_k) \setminus B = \mathrm{Sing}(\mathcal{C})$) and the Jacobian ideal of D_k is locally a complete intersection in every singular point of the pencil ($\mu_p(C_p) = \tau_p(C_p)$ for all $p \in \mathrm{Sing}(\mathcal{C})$). □

Remark. If the Jacobian ideal of D_k is not locally a complete intersection at p, then Z_k is not empty because $p \in \mathrm{supp}(Z_k)$, even if D_k contains all the singular curves.

This new hypothesis on the singularities must be added also in theorem 2.8 and in proposition 2.10. The proofs remain the same.

Theorem 2.8 *Assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth and that the Jacobian ideal of D^{sg} is locally a complete intersection. Assume also that D_k contains all the singular members of the pencil except the singular curves C_{α_i,β_i} for $i = 1,\dots,r$. Then,*

$$\mathcal{J}_{Z_k} = \mathcal{J}_{\nabla C_{\alpha_1,\beta_1}} \otimes \cdots \otimes \mathcal{J}_{\nabla C_{\alpha_r,\beta_r}}.$$

Proposition 2.10 *We assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth, that the Jacobian ideal of D^{sg} is locally a complete intersection and that D_k contains D^{sg}. Let C be a singular member in $\mathcal{C}(f,g)$ and Z its scheme of singular points. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{T}_{D_k} \longrightarrow \mathcal{T}_{D_k\setminus C} \longrightarrow \mathcal{J}_{Z/C}(n(3-k)-1) \longrightarrow 0,$$

where $\mathcal{J}_{Z/C} \subset \mathcal{O}_C$ defines Z into C.

References

[1] Schenck, H. and Tohăneanu, Ş. O. *Freeness of conic-line arrangements in* $\mathbb{P}^2$. Comment. Math. Helv. 84 (2009), no. 2, 235–258.

[2] Simis, A. and Tohăneanu, Ş. O. *Homology of homogeneous divisors*. Israel J. Math. 200 (2014), no. 1, 449–487

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