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PROPAGATIONS FROM A SPACE CURVE IN THREE SPACE WITH INDICATRIX A SURFACE

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Abstract. Generic singularities of rays emanating from a space curve in $\mathbb{R}^3$ in all directions with the rate determined by an indicatrix (independent of the point in $\mathbb{R}^3$) defined by a surface are classified. Similarly rays emanating from surface defined by an indicatrix given by a curve are also considered. Some applications to control theory are indicated.

1. Introduction

In this paper we solve two problems on the classification of local geometrical singularities that are related to control theory. We use some techniques from the singularity theory of caustics and wave fronts to study singularities of exponential mappings in a class of control problems which correspond to special integrable Hamiltonian systems with straight lines as extremals.

The first problem concerns a control system on a three-dimensional affine space with points $q \in \mathbb{R}^3$. We identify the tangent space $T_q\mathbb{R}^3$ with $\mathbb{R}^3$ itself. At each point q we choose an indicatrix I_q of admissible velocities $\dot{q} = \frac{\partial q}{\partial \mu}$ of motion which we assume is independent of the point q itself. Assume that this set is parametrised locally by a regular mapping $(x, y) \mapsto r_2(x, y)$ whose image is a surface M. We shall now write M in place of I_q.

An *admissible motion* is a smooth curve $\gamma(\mu) \in \mathbb{R}^3$, parametrised by a segment of the (affine) time axis μ, such that the velocity at each point $\dot{\gamma}$ belongs to the set of admissible velocities M.

Let $q_b(\mu)$ be the trajectory of an admissible motion of an initial point $b \in N$, issuing at $\mu = 0$ from an initial set N, where N is a space curve which is a submanifold in $\mathbb{R}^3$.

For a fixed value $\mu = \mu_0$ let $\mathcal{C}_b$ be the Banach manifold of all admissible trajectories defined on the segment $[0, \mu_0]$ from an initial point $b \in N$.

Consider the endpoint mapping $\mathcal{E}_b : \mathcal{C}_b \to \mathbb{R}^3$ which associates the endpoint $q_b(\mu_0)$ to a trajectory $q_b(\mu)$.

A corollary of the Pontryagin maximum principle, see [1, 9], states that critical values of $\mathcal{E}$ for all μ_0 trace extremal trajectories. In our case these are projections to $\mathbb{R}^3$ of solutions of the associated Hamiltonian canonical equations on the cotangent bundle

$$\dot{q} = \frac{\partial H_*(p, q)}{\partial p}, \quad \dot{p} = -\frac{\partial H_*(p, q)}{\partial q}.$$

Here the Hamiltonian function $H_*(p, q)$ on the cotangent bundle $T^*\mathbb{R}^3$ is the restriction (multivalued in general) to the subset $\{(p, q) \,|\, \exists (x, y) : \frac{\partial H(p,q,x,y)}{\partial x} = \frac{\partial H(p,q,x,y)}{\partial y} = 0\}$ of the function $H = \langle p, r_2(x, y)\rangle$, provided that the initial conditions (p_0, q_0) satisfy the relation $\langle p_0, v\rangle = 0$ for each vector v tangent to N at b. The angle brackets $\langle -, -\rangle$ denote the standard pairing of vectors $\mathbb{R}^3$ and covectors p of the dual space $(\mathbb{R}^3)^\wedge$.

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In our case extremals are straight lines (parametrised by $\mu \in \mathbb{R}$) $b + \mu v, b \in N, v \in M$, such that there is a covector p_0, which annihilates both tangent spaces T_bN and T_vM. Points on these lines with fixed μ form a wave front E_μ of the Legendre variety L_μ, which is the image of the Legendre submanifold L_0 of the initial conditions under the Hamiltonian flow.

The envelope $B(N, M)$ of these extremals is the union of singular points of sets of critical values of $\mathcal{E}_b$ for all b and μ_0.

Rephrasing the above in physical terms, consider an initial space curve in three space which emits rays from every point, and such that the speed of a ray at any point is completely determined by its direction. The *boundary of the set of attainability* of the rays after a given time μ will be the wave front E_μ. The caustics or focal sets correspond to the singularities of this set of attainability.

In this first problem as described above we consider an initial space curve with a velocity indicatrix defined by a surface. The classification where the indicatrix was also a space curve (independent of the point in $\mathbb{R}^3$) was given in [8]. The case of an initial surface and a velocity indicatrix described by a surface was studied in [3]. For completeness we also consider in the present paper a second problem interchanging the surface and the curve, i.e. the indicatrix is defined by a space curve and the initial manifold is a surface.

At present this second problem seems to have fewer applications than the first despite the fact that away from the initial surface the classification coincides with that of the first problem.

In the first problem the dimension of the indicatrix M is one less than the dimension of the ambient space $\mathbb{R}^3$, so the wave fronts E_μ form a family of equidistants in Finsler geometry. However as the dimension of the indicatrix in the second problem is not one less than the ambient space the wave fronts do not form a family of equidistants in Finsler geometry.

In this paper we classify the possible generic singularities of the envelopes $B(M, N)$ and of the family of wave fronts in both problems. We also classify the generic singularities near the initial surface itself in the second problem.

The method of classification of the singularities is similar to that of a related problem in [7]. In that paper the wave fronts were taken to be the closure of an affine ratio of pairs of points, one from a curve and the other from a surface that share parallel tangent planes. Here we consider in the first problem the surface, and then in the second problem the curve, to be at infinity. The computations were largely omitted from [7] and since in the present context they are slightly easier to write down we take this opportunity to include more details.

1.1. **Main definitions and results.** Let M be a smooth surface and let N be a smooth space curve both embedded in affine three space.

Consider a pair a, b of points $a \in M$ and $b \in N$ such that the plane tangent to the surface M at a is parallel to some plane tangent to N at b. The pair a, b is called a *parallel pair* and the straight line through a, b is called a *chord*. The envelope of the family of all chords is called the Minkowski set of M and N. In this paper we shall classify its generic singularities.

The chord $l(a, b)$ joining the parallel pair is defined by

$$l(a, b) = \{q \in \mathbb{R}^3 \mid q = \mu a + (1 - \mu)b, \mu \in \mathbb{R}\}. \tag{1}$$

The following definitions are valid for the propagating from the space curve case but similar definitions, by replacing μ with $1 - \mu = \lambda$ hold in the propagating from the surface case. The points which correspond to parallel tangent plane is the furthest point of the wave front from the curve. The wave front E_μ is the boundary of where the rays have reached at time μ. In the previous papers [3, 4, 7] barycentric coordinates were introduced on to the chords. Here however, we omit λ and just use the coordinate μ where $\mu = 0$ corresponds to the point on the curve N and $\mu = \infty$ corresponds to the point on the surface M.

A germ of the *affine μ-equidistant* E_μ of the pair (M,N) is the set of points $q \in \mathbb{R}^3$ such that $q = \mu a + b$ for given $\mu \in \mathbb{R}$ and for all parallel pairs (a,b) close to (a_0,b_0). Note that E_0 is the germ of N at b_0.

The space $\mathbb{R}^4_e = \mathbb{R} \times \mathbb{R}^3$ with coordinate $\mu \in \mathbb{R}$ (affine time), on the first factor is called the *extended affine space*. Denote by
$\rho : (\mu, q) \mapsto \mu$ the projection of $\mathbb{R}^4_e$ to the first factor and by $\pi : (\mu, q) \mapsto q$ the projection to the second factor.

The *affine extended wave front* $W(M,N)$ of the pair M,N is the union of all affine equidistants each embedded into its own slice of the extended affine space:

$$W(M,N) = \{(\mu, E_\mu)\} \subset \mathbb{R}^4_e.$$

The *bifurcation set* $B(M,N)$ of a family of affine equidistants (or of the family of chords) of the pair (M,N) is the image under π of the locus of the critical points of the restriction $\pi_r = \pi|_{W(M,N)}$. A point is *critical* if π_r at this point fails to be a regular projection of a smooth submanifold. In general $B(M,N)$ consists of two components: the *caustic* Σ is the projection of the singular locus of the extended wave front $W(M,N)$ and the *criminant* $\triangle$ is the (closure of the) image under π_r of the set of regular points of $W(M,N)$ which are the critical points of the projection π restricted to the regular part of $W(M,N)$. The caustic consists of the singular points of the momentary equidistants E_μ while the criminant is the envelope of the family of regular parts of the momentary equidistants. Besides being swept out by the momentary equidistants, the affine extended wave front is swept out by the liftings to $\mathbb{R}^{3+1}_e$ of chords. Each of them has a regular projection to the configuration space $\mathbb{R}^3$. Hence the bifurcation set $B(M,N)$ is essentially the envelope of the family of chords.

In the generic setting the distinguished chords split into three distinct sub-cases: In the first (*transversal*) case the base points $a_0 \in M$ and $b_0 \in N$ are distinct and the chord through them is transversal to both M and N. In the second (*tangential*) case the base points $a_0 \in M$ and $b_0 \in N$ are distinct but the tangent line to N lies in the tangent plane to M. A subcase of the tangential case called the *supertangential* case occurs when the line tangent to the curve N at b contains the point a, i.e. the chord and the tangent line are the same.

Definition 1.1. Two germs of families F_1 and F_2 in parameters μ, q are called *space-time contact equivalent* if there exists a nonzero function $\phi(z,\mu,q)$ and diffeomorphism $\widehat{\theta} : (z,\mu,q) \mapsto (Z(z,\mu,q), P(\mu,q), Q(q))$ such that $F_1 = \phi F_2 \circ \widehat{\theta}$.

Notice that the diffeomorphism $\widehat{\theta} : (\mu,q) \mapsto (P(\mu,q), Q(q))$ of the total parameter space $\mathbb{R}^{3+1}$ maps the extended wave front of the first family to the extended wave front of the second family and the diffeomorphism $\widehat{\theta} : q \mapsto Q(q)$ of the q-parameter space $\mathbb{R}^3$ maps the bifurcation set of the first family to the bifurcation set of the second family.

Definition 1.2. Two germs of families F_1 and F_2 are called *time-space contact equivalent* if there exists a nonzero function $\phi(z,\mu,q)$ and diffeomorphism $\widetilde{\theta} : (z,\mu,q) \mapsto (Z(z,\mu,q), P(\mu), Q(\mu,q))$ such that $F_1 = \phi F_2 \circ \widetilde{\theta}$.

The diffeomorphism $\widetilde{\theta} : (\mu,q) \mapsto (P(\mu), Q(\mu,q))$ preserves the fibration of the μ,q space into fibres parallel to the q space. If two families are time-space contact equivalent then their respective families of momentary wave fronts are diffeomorphic.

The main results are as follows: The first theorem which concerns the wave fronts follows immediately from the results of [7].

Theorem 1.3. *The families of wave fronts and their bifurcations in the propagating from the curve and in the propagating from the surface cases are diffeomorphic to those in the affine ratio case* [7]*. In fact the generating functions are time-space contact equivalent.*

The following theorems all concern the projection π and are related to the caustics. The theorems are the complete classification of generic singularities in the various settings. The list is the same as in [7]. Unlike theorem 1.3 they do not follow immediately from the previous papers and require separate calculation.

Theorem 1.4. *In the propagating from the curve transversal case outside M and N the germ at any point of the envelope of the family of chords for generic M and N is diffeomorphic to one of the standard caustics of A_k type with $k = 2, 3$ or 4 (regular surface, cuspidal edge or swallowtail).*

Theorem 1.5. *In the propagating from the curve tangential case the germ at any point outside M and N of the envelope of the family of chords for generic N and M is diffeomorphic to one of the standard caustics of the boundary singularities of the types B_2, B_3, B_4, C_3, C_4 or F_4. If moreover the tangent line to the curve coincides with the chord (supertangential case) then generically only B_2 and C_3 occur.*

Theorem 1.6. *In the propagating from the surface transversal case outside M and N the germ at any point of the envelope of the family of chords for generic M and N is diffeomorphic to one of the standard caustics of A_k type with $k = 2, 3$ or 4 (regular surface, cuspidal edge or swallowtail).*

Theorem 1.7. *In the propagating from the surface transversal case the envelope of the family of chords transversally intersects the surface M when $\lambda = 0$ generically at either its smooth points or at points of a cuspidal ridge.*

Theorem 1.8. *In the propagating from the surface tangential setting the germ at any point outside M and N of the envelope of the family of chords for generic N and M is diffeomorphic to one of the standard caustics of the boundary singularities of the types B_2, B_3, B_4, C_3, C_4 or F_4. If moreover the tangent line to the curve coincides with the chord (supertangential case) then generically only B_2 and C_3 occur.*

1.2. **Generating families.** Now consider the following generating family $\mathcal{F}_1$ in the propagating from the curve case. The family has variables $n \in (\mathbb{R}^3)^\wedge \setminus \{0\}$, t and (x, y), and parameters $(\mu, q) \in \mathbb{R} \times \mathbb{R}^3$;

$$\mathcal{F}_1(n, t, x, y, \mu, q) = \langle r_1(t) + \mu r_2(x, y) - q, n\rangle \tag{2}$$

where $r_1(t)$ is the embedding with the image N, and $r_2(x, y)$ is the embedding with image M.

In the propagating from the surface case we use the generating family

$$\mathcal{F}_2(n, t, x, y, \lambda, q) = \langle \lambda r_1(t) + r_2(x, y) - q, n\rangle \tag{3}$$

with the same variables as $\mathcal{F}_1$ but parameters $(\lambda, q) \in \mathbb{R} \times \mathbb{R}^3$.

In the paper [7] the affine ratio case was studied and the generic bifurcations of the wave fronts were classified. There the generating family used was

$$\widetilde{\mathcal{F}}(n, t, x, y, \mu, q) = \langle (1 - \mu) r_1(t) + \mu r_2(x, y) - q, n\rangle. \tag{4}$$

We now show that the two family germs $\widetilde{\mathcal{F}}$ and $\mathcal{F}_1$ are time-space equivalent (see theorem 1.3) and hence the classification of their generic wave fronts and their bifurcations are in fact the same.

Proof of theorem 1.3.
Assuming that $\mu \neq 1$ we can divide the family (4) by $(1-\mu)$ to give

$$\widehat{\mathcal{F}}(n,t,x,y,\mu,q) = \langle r_1(t) + \frac{\mu}{1-\mu} r_2(x,y) - \frac{q}{1-\mu}, n\rangle.$$

This is time-space contact equivalent to

$$\widehat{\mathcal{F}_1} = \langle r_1(t) + \tilde{\mu} r_2(x,y) - \tilde{q}, n\rangle$$

with $\tilde{\mu} = \frac{\mu}{1-\mu}$ and $\tilde{q} = -\frac{q}{1-\mu}$. $\square$

If $\mu = 1$ then this is equivalent to the present case at infinity and so does not appear in this “non-projective" setting. Similar considerations show that the family $\widetilde{\mathcal{F}}$ is time-space contact equivalent to the family $\mathcal{F}_2$.

2. Propagating from the curve in the transversal setting

In the transversal setting up to an appropriate affine transformation of $\mathbb{R}^3$ we can always assume that in some coordinate system (x,y,z) the base parallel pair a_0, b_0 coincides with the pair of points $(0,0,-1), (0,0,0)$, the tangent plane to the surface M at a_0 is parallel to the (x,y)-coordinate plane and the tangent line to the curve N at b_0 coincides with the x-axis.

In these coordinates the surface M in the neighbourhood of a_0 is the graph $M = \{(x,y,z) | z = f(x,y) - 1)$ of the function f with vanishing 1-jet. Let $f(x,y) = \sum_{i+j\geq 2} f_{ij} x^i y^j$ be the Taylor decomposition of the germ of f at the origin. Define the curve N to be the set $\{(t, \alpha(t), \beta(t))\}$ with the functions $\alpha(t) = \alpha_2 t^2 + \alpha_3 t^3 + ...$ and $\beta(t) = \beta_2 t^2 + \beta_3 t^3 + ...$ starting with at least quadratic terms in t.

Proposition 2.1. *The germ of the family $\mathcal{F}_1$ at a point corresponding to a point on the base chord is stably-equivalent to the product of the family germ $\Phi(t,\mu,q) = \beta(t) + \mu[f(\widehat{x},\widehat{y}) - 1] - q_3$ at the subset $\widehat{S_0} = \{t = 0, q_1 = q_2 = 0\}$ with a nonzero factor. Here we use the substitution $\widehat{x} = \frac{q_1 - t}{\mu}, \widehat{y} = \frac{q_2 - \alpha(t)}{\mu}$.*

Proof. Writing the family $\mathcal{F}_1$ in the coordinate form we get

$$\mathcal{F}_1 = An_1 + Bn_2 + Cn_3$$

where

$$A = t + \mu x - q_1$$
$$B = \alpha(t) + \mu y - q_2$$

and

$$C = \beta(t) + \mu[f(x,y) - 1] - q_3$$

For $\mu \neq 0$ the functions A and B are regular and we can choose A, B as the coordinate functions instead of x and y. In particular we can write $x = \frac{A + q_1 - t}{\mu}$ and $y = \frac{B + q_2 - \alpha(t)}{\mu}$.

So in the new coordinates we have $\mathcal{F}_1 = An_1 + Bn_2 + C(A,B,t,\mu,q)n_3$. The function C does not depend on n_1 and n_2 and the Hadamard lemma implies $C(A,B,t,\mu,q) = C(0,0,\mu,t,q) + A\varphi_1 + B\varphi_2$, where φ_1 and φ_2 are smooth functions in A, B, t, μ and q which vanish at the origin.

Now the function $\mathcal{F}_1$ takes the form $\mathcal{F}_1 = A(n_1 + \varphi_1 n_3) + B(n_2 + \varphi_2 n_3) + C(0,0,t,\mu,q)$ where the first two terms represent a non degenerate quadratic form in the independent variables $A, (n_1 + \varphi_1 n_3), B$ and $(n_2 + \varphi_2 n_3)$ in the vicinity of the point on the base chord.

Therefore, the function $\mathcal{F}_1$ is stably-equivalent to the function $\Phi = C(0,0,t,\mu,q)$ being the restriction of the function C to the subspace $A = B = 0$. So to study the envelope of chords and the families of wave fronts we can study the family germ

$$\Phi(t,\mu,q) = \beta(t) + \mu\left[f\left(\frac{q_1 - t}{\mu}, \frac{q_2 - \alpha(t)}{\mu}\right) - 1\right] - q_3 \quad \square.$$

For the family germ Φ at the point $m_0 = (0,\mu_0,0,0,q_3 = -\mu_0)$, on the base chord $l(a_0,b_0)$ denote by $g(t) = \Phi(t,\mu_0,0,0,q_3)$ at t the respective organising centre function.

To determine the singularity type of the generating family germ Φ at the point m_0 and the respective versality conditions denote by $\Phi_k(\mu,q)$ the coefficients at t^k in the Taylor decomposition of Φ with respect to t at the origin.

$$\Phi = \Phi_0 + \Phi_1 t + \Phi_2 t^2 + \Phi_3 t^3 + \Phi_4 t^4 + \Phi_5 t^5 + ...$$

The first few formulas where terms of second order or greater in q_1 and q_2 are denoted by dots are as follows:

$$\begin{aligned}
\Phi_0 &= -\mu - q_3 \\
\Phi_1 &= \frac{1}{\mu}(-2f_{20}q_1 + f_{11}q_2) + ... \\
\Phi_2 &= \beta_2 + \frac{1}{\mu}(f_{20} + \alpha_2 f_{11}q_1 - 2\alpha_2 f_{02}q_2) + \frac{1}{\mu^2}(3f_{30}q_1 + f_{21}q_2) + ... \\
\Phi_3 &= \beta_3 + \frac{1}{\mu}(\alpha_2 f_{11} - \alpha_3 f_{11}q_1 - 2\alpha_3 f_{02}q_2) + \frac{1}{\mu^2}(-f_{30} + 2\alpha_2 f_{21}q_1 + 2\alpha_2 f_{12}q_2) \\
&\quad + \frac{1}{\mu^3}(-4f_{40}q_1 - f_{31}q_2) + ...
\end{aligned}$$

Setting in these formulas $q_1 = q_2 = 0$ we get the expressions of the Taylor coefficients of the organising centre $g_k = \Phi_k|_{q_1=q_2=0}$ at a chord point m_0.

2.1. **Normal forms of the Minkowski set.** The following proposition together with explicit calculations from the normal forms prove theorem 1.4.

Proposition 2.2. *For a generic pair of M and N at any point q of a base chord (a_0,b_0) except the point b_0 itself ($\mu = 0$) the germ of the respective generating family Φ is space-time contact equivalent to one of the standard versal deformations in parameters $(\mu,q) \in \mathbb{R} \times \mathbb{R}^3$ of the function germs at the origin in the variable t of the type A_k for $k = 1,\dots,4$ as follows:*

$$\begin{aligned}
A_1 : \Phi &= t^2 + \mu; & A_2 &: \Phi = t^3 + q_1 t + \mu; \\
A_3 : \Phi &= t^4 + q_2 t^2 + q_1 t + \mu; & A_4 &: \Phi = t^5 + q_3 t^3 + q_2 t^2 + q_1 t + \mu.
\end{aligned}$$

2.2. **Recognition of transversal singularities.** If β_2 is nonzero, that is the base tangent plane is not the osculating plane to the curve N then we always get a unique A_2 singularity at the point $\mu_c = -\frac{f_{20}}{\beta_2}$. If however $\beta_2 = 0$ then no caustic point occurs on the chord unless additionally $f_{20} = 0$ in which case the whole chord is of type A_2 and therefore belongs to the caustic. These are isolated chords and the situation when these occur at $f_{20} = \beta_2 = 0$ is called the *flattening case*.

If the condition $\beta_3 = \frac{f_{30}\beta_2^2}{f_{20}^2} + \frac{\alpha_2 f_{11}\beta_2}{f_{20}}$ holds then the caustic point at μ_c will be of the type A_3. If in addition to the condition for an A_3 singularity the condition $\beta_4 = \frac{f_{40}{\beta_2^3}^3}{f_{20}} + \frac{\alpha_2 f_{21}{\beta_2}^2}{{f_{20}}^2} + \frac{\alpha_3 f_{11}\beta_2}{f_{20}} + \frac{{\alpha_2}^2 f_{02}\beta_2}{f_{20}}$ also holds, together with g_5 being nonzero, then the caustic point at μ_c will be of the type A_4. The singularities of this type occur at isolated points due to genericity.

In the flattening case $f_{20} = \beta_2 = 0$, in addition to the whole chord being of type A_2, there also exist two points where A_3 singularities occur at $\mu = \frac{-f_{11}\alpha_2 \pm \sqrt{f_{11}^2\alpha_2^2 + 4\beta_3 f_{30}}}{2\beta_3}$.

Proof of proposition 2.2

The proof of the proposition uses the property that $\frac{\partial\Phi}{\partial\lambda} \neq 0$ which holds in the transversal case. The stability with respect to space-time contact equivalence of the germ Φ with this property coincides with its stability with respect to standard contact equivalence. Therefore to show stability with respect to space-time contact equivalence we proceed by proving that each singularity in turn is generically versal with respect to standard contact equivalence, (see [2]).

Let $\Phi : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^3 \mapsto \mathbb{R}$ be an unfolding of a function $g(t), t \in \mathbb{R}$ with parameters $\mu, q \in \mathbb{R}\times\mathbb{R}^3$ and let $g(t)$ have an A_k singularity at the origin.

Denote by δ_{ij} the coefficients of the k-jet of Φ at the origin where

$$\delta_{i1} = \frac{\partial^{i+1}\Phi}{\partial t^i \partial q_3}, \delta_{i2} = \frac{\partial^{i+1}\Phi}{\partial t^i \partial \mu}, \delta_{i3} = \frac{\partial^{i+1}\Phi}{\partial t^i \partial q_1} \text{ and } \delta_{i4} = \frac{\partial^{i+1}\Phi}{\partial t^i \partial q_2}.$$

The jet matrix for the family of functions Φ shall be denoted M_4 and let M_k with $k \leq 4$ be the matrix consisting of the first k rows of M_4. We only consider $k \leq 4$ due to genericity.

The matrix $M_4 = (\delta_{ij})$ up to a factor of the rows for μ nonzero is given by

$$M_4 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -2f_{20} & -f_{11} \\ 0 & \delta_{32} & \delta_{33} & \delta_{34} \\ 0 & \delta_{42} & \delta_{43} & \delta_{44} \end{pmatrix}$$

where

$$\begin{aligned} \delta_{32} &= -f_{20}, \quad \delta_{33} = -\alpha_2 f_{11}\mu + 3f_{30}, \quad \delta_{34} = -2\alpha_2 f_{02}\mu + f_{21}, \\ \delta_{42} &= -\alpha_2 f_{11}\mu + 2f_{30}, \quad \delta_{43} = -\alpha_3 f_{11}\mu^2 + 2\alpha_2 f_{21}\mu - 4f_{40}, \quad \delta_{44} = -2\alpha_3 f_{02}\mu^2 + 2\alpha_2 f_{12}\mu - f_{31}. \end{aligned}$$

Then function Φ is right-versal if and only if the matrix M_k has rank k. Notice that the conditions $g_1 = 0, ..., g_k = 0$ define a Whitney stratification in the jet space of the embeddings. In fact each of these conditions outside λ being zero defines a regular hyper-surface in the space of germs and moreover those hyper-surfaces are mutually transversal since each equation $g_i = 0$ involves only one variable β_i and can be solved for it in terms of coefficients f_{jl} and α_s.

Versality of an A_1 singularity

The proof is immediate because the matrix

$$M_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix}$$

always has the maximal rank of 1. $\square$

Versality of an A_2 singularity

The versality of the A_2 singularities is determined by whether the matrix M_2 has maximal rank 2 where

$$M_2 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -2f_{20} & -f_{11} \end{pmatrix}.$$

The matrix M_2 has non-maximal rank only if both f_{11} and f_{20} vanish. If f_{20} vanishes and $\beta_2 \neq 0$ then recall the caustic occurs at $\mu = 0$ on the curve itself. If $f_{20} = \beta_2 = 0$ then the whole chord belongs to the caustic. In this case the vanishing of β_2, f_{20} and f_{11} is non-generic. Therefore, away from the curve and surface, A_2 singular points at μ_c are versal. $\square$

Versality of an A_3 singularity
If $f_{20} \neq 0$ then clearly the minor consisting of the first three columns of M_3 as nonzero determinant. In the flattening case $f_{20} = \beta_2 = 0$ the derivative matrix M_3 takes the form

$$M_3 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & f_{11} \\ 0 & 0 & -\mu\alpha_2 f_{11} + 3f_{30} & -2\mu\alpha_2 f_{02} + f_{21} \end{pmatrix}$$

and recall that the A_3 singularities occur at $\mu = \frac{-\alpha_2 f_{11} \pm \sqrt{\alpha_2^2 f_{11}^2 - 4\beta_3 f_{30}}}{2\beta_3}$. For this value of μ the determinant of both of the minors of M_3 vanishes if either $f_{11} = 0$ or $\beta_3 = -\frac{4{f_{11}}^2{\alpha_2}^2}{9f_{30}}$. Neither of these conditions holds generically,so generically A_3 singularities are versal.

Versality of an A_4 singularity.
An A_4 singularity occurs on the base chord at the point m_0 when $g_2 = g_3 = 0$ and

$$g_4 = \beta_4 + \frac{1}{\mu}(\alpha_3 f_{11} + \alpha_2^2 f_{02}) - \frac{1}{\mu^2}\alpha_2 f_{21} + \frac{1}{\mu^3} f_{40} = 0,$$

but $g_5 \neq 0$. Notice also that A_4 cannot happen in the flattening case due to genericity.

The versality of A_4 singularities holds if the determinant

$$\det(M_4) = 2f_{20}[\delta_{32}\delta_{44} - \delta_{34}\delta_{42}] - f_{11}[\delta_{32}\delta_{43} - \delta_{33}\delta_{42}]$$

is nonzero. Since generically A_4 singularities cannot occur in the flattening case we assume that β_2 and f_{20} are nonzero. The condition that the determinant is zero can be solved for f_{31} as a function of the other terms.

The codimension of the stratum which corresponds to an A_4 singularity together with the vanishing of $\det(M_4)$ is greater than 3 so A_4 singularities are generically versal. This completes the proof of proposition 2.2. Explicit calculations from the normal forms completes the proof of theorem 1.4. $\square$

2.3. **Propagating from the space curve in the tangential setting.** In the tangential case we use the same family as was used in the transversal case

$$\mathcal{F}_1(n, t, x, y, \mu, q) = \langle r_1(t) + \mu r_2(x, y) - q, n\rangle.$$

Here we assume that the base chord lies in the plane tangent to M at a_0 (tangential setting). If the base chord and the tangent line to the curve N at b_0 are not collinear then in some coordinate system (x, y, z) the base points a_0, b_0 coincide with the points $(0, 1, 0), (0, 0, 0)$, the curve N at the origin is tangent to the x-axis and the tangent plane to the surface M coincides with the (x, y)-coordinate plane. Now the surface M is defined by the embedding

$$r_2 : U \to \mathbb{R}^3,\ r_2(x, y) = (x, y + 1, f(x, y)),\ (x, y) \in U \subset \mathbb{R}^2$$

where the function $f(x, y)$ has zero 1-jet, and the curve N is defined by the embedding

$$r_1 : V \to \mathbb{R}^3,\ r_1 : t \mapsto (t, \alpha(t), \beta(t)),\ t \in V \subset \mathbb{R}$$

of some neighbourhood V of the origin in $\mathbb{R}$ where $\alpha(t)$ and $\beta(t)$ start with second order terms.

After an appropriate stabilisation the initial generating family germ $\mathcal{F}_1$ at the point $\mu = \mu_0, t = 0, q = 0$ reduces to the form

$$\Phi(t, \varepsilon, q) = \beta(t) + (\mu_0 - \varepsilon) f\left(\frac{q_1 - t}{\mu_0 - \varepsilon}, \frac{\widetilde{q_2} - \alpha(t) - \varepsilon}{\mu_0 - \varepsilon}\right) - q_3 \tag{5}$$

where $\varepsilon = \mu_0 - \mu$ varies in the vicinity of the origin and $\widetilde{q_2} = q_2 - \mu_0$.

Consider the organising centre $g(t,\varepsilon) = \Phi|_{q_1=\widetilde{q_2}=q_3=0}$ of the family and decompose it as $g(t,\varepsilon) = \sum_{i+j\geq 2} a_{ij}t^i\varepsilon^j$ where the first few terms are:

$$\begin{aligned}
a_{20} &= \beta_2 + \frac{1}{\mu_0}f_{20}, \quad a_{11} = -\frac{1}{\mu_0}f_{11}, \quad a_{02} = \frac{1}{\mu_0}f_{02},\\
a_{30} &= \beta_3 + \frac{1}{\mu_0}\alpha_2 f_{11} - \frac{1}{\mu_0^2}f_{30},\\
a_{21} &= -\frac{2}{\mu_0}\alpha_2 f_{02} + \frac{1}{\mu_0^2}(f_{20}+f_{21}),
\end{aligned}$$

The space-time contact equivalence of the families of type Φ corresponds to *fibred contact equivalence* of the respective organising centres $g(t,\varepsilon)$: diffeomorphisms of the form $(t,\varepsilon) \to (\hat{t}(t,\varepsilon), \hat{\varepsilon}(\varepsilon))$ and multiplications by nonzero functions act on g.

The well-known Arnold-Goryunov low dimensional fibred contact classification (which coincides with simple boundary classes) provides all generic space-time contact stable families depending on three parameters (here $k = 2, 3$ or 4):

$$\begin{aligned}
B_k: &\quad \pm t^2 + \varepsilon^k + q_{k-2}\varepsilon^{k-2} + \ldots + q_3,\\
C_2 \approx &\quad B_2,\\
C_3: &\quad t^3 + t\varepsilon + q_1\varepsilon + q_3,\\
C_3: &\quad t^4 + t\varepsilon + q_2 t^2 + q_1\varepsilon + q_3,\\
F_4: &\quad t^3 + \varepsilon^2 + q_2 t\varepsilon + q_1 t + q_3.
\end{aligned} \tag{6}$$

The proof of theorem 1.5 consists of checking the versality and genericity conditions for germs of the family Φ.

Singularities of the type B_k occur $a_{20} \neq 0$. When the quadratic form of $g(t,\varepsilon)$ is non-degenerate then the singularity of type B_2 occurs. If the quadratic form is degenerate, that is $4a_{20}a_{02} - a_{11}^2 = 0$, the singularity is of type B_3. This occurs when $\mu_0 = \frac{f_{11}^2 - 4f_{20}f_{02}}{4\beta_2 f_{02}}$ so every chord in the tangential setting has a singularity of type B_3 (which may occur on the curve or at infinity). The B_3 singularity can become more degenerate at isolated points to form the B_4 type. This condition can be solved for β_3 as a function of the other terms. Any further degenerations are excluded due to genericity.

The C_3 singularity occurs when $a_{20} = 0$ and both $a_{20} \neq 0$ and $a_{11} \neq 0$. This happens at a single point on the chord when $\mu_0 = -\frac{f_{20}}{\beta_2}$. This can become more degenerate if $a_{30} = 0$ and $a_{40} \neq 0$ to form the singularity of type C_4. The singularity of type F_4 belongs to the intersection of the closures of the B_3 and C_3 singularities and occurs when $\mu_0 = -\frac{f_{20}}{\beta_2}$ and $f_{11} = 0$. Similar considerations using slightly different embeddings show that in the supertangential case, away from the curve and surface, only singularities of type B_2 and C_3 occur generically.

3. Propagating from the surface in the transversal setting

We now turn our attention to the case where our initial starting manifold is a surface and the indicatrix of admissible velocities at each point defined by a space curve. Recall that in this case we use the generating function

$$\mathcal{F}_2(n,t,x,y,\lambda,q) = \langle \lambda r_1(t) + r_2(x,y) - q, n\rangle$$

where as before $r_1(t)$ is the embedding with the image of the space curve N, and $r_2(x,y)$ is the embedding with image the surface M. In this case we choose affine coordinates so that near a distinguished chord the surface is at the origin and the tangent plane is the (x,y)-coordinate plane, and the space curve contains the point $(0,0,1)$ and has tangent vector in the direction of the x-axis.

Proposition 3.1. *The family germ $\mathcal{F}_2$ at a point corresponding to a point on the base chord is stably-equivalent to the product of the family germ $\Phi(t,\mu,q) = \lambda(\beta(t)+1) + [f(\widehat{x},\widehat{y})] - q_3$ at the subset $\widehat{S_0} = \{t = x = y = 0, q_1 = q_2 = 0\}$ with a nonzero factor. Here we use the substitution $\widehat{x} = q_1 - \lambda t, \widehat{y} = q_2 - \lambda\alpha(t)$.*

Proof. Writing the family $\mathcal{F}_2$ in the coordinate form we get

$$\mathcal{F} = An_1 + Bn_2 + Cn_3$$

where

$$\begin{aligned} A &= \lambda t + x - q_1 \\ B &= \lambda\alpha(t) + y - q_2 \\ C &= \lambda(\beta(t)+1) + f(x,y) - q_3 \end{aligned}$$

As in the previous case we make an appropriate substitution, this time $x = q_1 - \lambda t$ and $y = q_2 - \lambda\alpha(t)$, and use the Hadamard lemma to show that this is stably equivalent to the family

$$\Phi(t,\lambda,q) = \lambda\beta(t) + \lambda + [f\,(q_1 - \lambda t, q_2 - \lambda\alpha(t))] - q_3.$$

Expanding the function Φ as a Taylor decomposition with respect to t at the origin where $\Phi = \Sigma_{n=0}^{\infty}\Phi_k t^k$, up to linear terms in q_1 and q_2, has the first few coefficients:

$$\begin{aligned} \Phi_0 &= \lambda - q_3 \\ \Phi_1 &= -\lambda(2f_{20}q_1 + f_{11}q_2) \\ \Phi_2 &= \lambda(\beta_2 - f_{11}\alpha_2 q_1 - 2f_{02}\alpha_2 q_2 + \lambda f_{20} + 3\lambda f_{30}q_1 + f_{21}\lambda q_2) \end{aligned}$$

Setting in these formulas $q_1 = q_2 = 0$ we get the expressions of the Taylor coefficients of the organising centre $g_k = \Phi_k|q_1 = q_2 = 0$ at a chord point m_0.

As with the propagating from the space curve case away from the initial starting manifold an A_2 singularity occurs on the chord at $\lambda_c = \frac{-\beta_2}{f_{20}}$. This becomes more degenerate as an A_3 singularity if additionally $\beta_3 = \frac{{\beta_2}^2 f_{30}}{{f_{20}}^2} + \frac{\beta_2 f_{11}\alpha_2}{f_{20}}$ and type A_4 if also $\beta_4 = -\lambda^3 f_{40} + \lambda^2\alpha_2 f_{21} - \lambda\alpha_3 f_{11} - \lambda\alpha_2^2 f_{02}$. Notice that these conditions for the singularity to be more degenerate are the same as those in the propagating from the space curve case. In the flattening case when $f_{20} = \beta_2 = 0$ the whole chord belongs to the caustic and is type A_2 everywhere except two points $\lambda = \frac{\alpha_2 f_{11} \pm \sqrt{{\alpha_2}^2 {f_{11}}^2 + 4\beta_3 f_{30}}}{2f_{30}}$ where singularities of type A_3 occur.

Consider the derivative matrix given by

$$M_4 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -2f_{20} & -f_{11} \\ 0 & \delta_{32} & \delta_{33} & \delta_{34} \\ 0 & \delta_{42} & \delta_{43} & \delta_{44} \end{pmatrix}$$

where

$$\begin{aligned} \delta_{32} &= 2\lambda f_{20} + \beta_2, \quad \delta_{33} = 3\lambda^2 f_{30} - \lambda\alpha_2 f_{11}, \quad \delta_{34}\lambda^2 f_{21} - 2\lambda\alpha_2 f_{02}, \\ \delta_{42} &= -3\lambda^2 f_{30} + 2\lambda\alpha_2 f_{11} + \beta_3, \quad \delta_{43} = -4\lambda^3 f_{40} + 2\lambda^2\alpha_2 f_{21} - \lambda\alpha_3 f_{11}, \\ \delta_{44} &= -\lambda^3 f_{31} + 2\lambda^2\alpha_2 f_{12} - 2\lambda\alpha_3 f_{02}. \end{aligned}$$

We can use the same arguments as we used in the propagating from the space curve case to show that all the generic A_k singularities are versal.

3.1. **Propagating from the surface in the tangential setting.** Assume that the base chord is in the tangential setting, that is it lies in the plane tangent to M at a_0 but it is not collinear with the tangent line to the curve N at b_0. In some coordinate system the surface contains the origin at a_0 and the tangent plane at this point is the (x, y)-coordinate plane, the curve passes through the point $(0, 1, 0)$ at b_0 and the tangent line has the same direction as the x-axis. Using appropriate embeddings $r_1(t)$ and $r_2(x, y)$ the generating $\mathcal{F}_2$ can be expanded as a vector to give

$$\mathcal{F}_2 = (x + \lambda t - q_1)n_1 + (y + \lambda(\alpha(t) + 1) - q_2)n_2 + ((f(x, y) + \lambda\beta(t) - q_3)n_3.$$

Proposition 3.2. *Using the substitution $x = q_1 - \lambda t$, $y = q_2 - \lambda(\alpha(t) + 1)$ the family $\mathcal{F}_2$ at the point $\lambda = \lambda_0, t = 0, q = 0$ is stably equivalent to the family*

$$\Phi(t, \varepsilon, q) = (\lambda_0 + \varepsilon)\beta(t) + f(q_1 - (\lambda_0 + \varepsilon)t, \widetilde{q_2} - \lambda_0\alpha(t) - \varepsilon\alpha(t) - \varepsilon). \tag{7}$$

where $\lambda = (\lambda_0 + \varepsilon)$ varies in the vicinity of the origin and $q_2 = \widetilde{q_2} + \lambda_0$.

Consider the organising centre $g(t, \varepsilon) = \Phi|_{q_1=\widetilde{q_2}=q_3=0}$ of the family and decompose it as $g(t, \varepsilon) = \sum_{i+j\geq 2} a_{ij}t^i\varepsilon^j$ where the first few terms are

$$\begin{aligned}
a_{20} &= f_{20}\lambda_0^2 + \lambda_0\beta_2, \quad a_{11} = f_{11}\lambda_0, \quad a_{02} = f_{02},\\
a_{30} &= \lambda_0\beta_3 + f_{11}\lambda_0^2\alpha_2 - f_{30}\lambda_0^3,\\
a_{21} &= \beta_2 + 2f_{02}\lambda_0\alpha_2 - f_{21}\lambda_0^2 + 2f_{20}\lambda_0,\\
a_{12} &= f_{11} - f_{12}\lambda_0, \quad a_{03} = -f_{03}.
\end{aligned}$$

The list of generic singularities coincides with the list (6).

The whole of each chord in the tangential setting is of type at B_2 except for at most two points which can be more degenerate. Generically each chord will consist of a B_3 singularity and a C_3 singularity. At isolated chords one of these can be more degenerate to form either B_4 or C_4. Also at isolated chords the B_3 and C_3 singularities can occur at the same point to give an F_4 singularity.

When the quadratic form is degenerate, that is $4a_{20}a_{02} - a_{11}^2 = 0$, a singularity of type B_3 occurs. This happens at $\lambda = \frac{4f_{02}\beta_2}{f_{11}^2 - 4f_{02}f_{20}}$. For isolated chords one of these can be more degenerate giving the singularity of type B_4.

This condition can be solved for f_{03} as a function of the other terms.

Singularities of type C_3 occur at $\lambda = -\frac{\beta_2}{f_{20}}$. This can be more degenerate to form a C_4 singularity if $\beta_3 = \frac{\beta_2^2 f_{30}}{f_{20}^2} + \frac{\beta_2 f_{11}\alpha_2}{f_{20}}$. An F_4 singularity will result if both the conditions for a C_3 and a B_3 singularity occur, namely if $\lambda = -\frac{\beta_2}{f_{20}}$ and $f_{11} = 0$. Further degenerations are excluded due to genericity.

Checking the versality and genericity conditions for germs of the family Φ completes the proof of Theorem 1.8.

Similar considerations using different embeddings show that in the supertangential case, away from the curve and surface, generically only singularities of type B_2 and C_3 occur. □

3.2. **Propagating from the surface in the transversal case in the vicinity of the surface.** Up until now we have assumed that λ is nonzero and have classified the singularities away from the surface and space curve. In this section we study the generic caustic near the surface itself, that is when λ is close to zero. We use the standard generating family in the propagating from the surface case $\mathcal{F}_2$ and proposition 3.1 implies the generating family is stably equivalent to

$$\Phi(t,\lambda,q)=\lambda(\beta(t)+1)+f(q_1-\lambda t,q_2-\lambda\alpha(t))-q_3.$$

This can be written as

$$\Phi=\lambda\left(\frac{f-f_0}{\lambda}+\beta+1\right)+f_0-q_3$$

where $f_0=f|_{q_1=q_2=0}$ and $\frac{f-f_0}{\lambda}$ is smooth. Introduce the new parameter $\widetilde{q_3}=-q_3+f_0$ which vanishes on the surface, yielding

$$\Phi=\lambda\left(\frac{f-f_0}{\lambda}+\beta+1\right)+\widetilde{q_3}.$$

Denote by $\Phi_0,\Phi_1,...$ the terms of the power series decomposition in λ of the contents of the brackets. With terms of order greater than 4 in t or greater than 1 in q_1 and q_2 denoted by dots the generating function Φ is written

$$\mathcal{F}=\widetilde{q_3}+\lambda\left(\Phi_0+...+\lambda(\Phi_1+...)+\lambda^2(\Phi_2+...)+...\right) \tag{8}$$

where

$$\Phi_0=1+\beta(t)+(-2f_{02}\alpha(t)-f_{11}t)\,q_2+(-2f_{20}t-f_{11}\alpha(t))\,q_1$$

and

$$\Phi_1=f_{11}t\alpha(t)+f_{20}t^2+f_{02}\alpha(t)^2+\\ \left(2f_{12}t\alpha(t)+f_{21}t^2+3f_{03}\alpha(t)^2\right)q_2+\left(2f_{21}t\alpha(t)+f_{12}\alpha(t)^2+3f_{30}t^2\right)q_1.$$

Proposition 3.3. *The family germ $\mathcal{F}$ can be written in the form $\mathcal{F}=\lambda+\widetilde{q_3}H(t,q_1,q_2,\widetilde{q_3})$ where the lower degree terms with respect to $\widetilde{q_3}$ of function H are: $H=\frac{1}{\Phi_0}+\frac{\Phi_1\widetilde{q_3}}{\Phi_0^3}+...$*

Lemma 3.4. *Assume $H(t,q_1,q_2,\widetilde{q_3})$ is $\mathcal{R}^+$-versal with respect to q_1 and q_2 only; then the family germ $\mathcal{F}=\lambda+\widetilde{q_3}H$ is space-time stable with respect to deformations inside the space $W=\lambda+\widetilde{q_3}\widetilde{H}(t,q1,q2,\widetilde{q_3})$ such that $\frac{\partial W}{\partial\widetilde{q_3}}\neq 0$.*

Proposition 3.5. *For generic curve and surface germs in the transversal setting the function $H(t,\lambda,q)$ is versal for standard $\mathcal{R}^+$-equivalence with respect to q_1 and q_2 only.*

The first few terms of the Taylor decomposition of H with respect to t at the origin, namely $H=\Sigma_{k=0}H_k(t,q)t^k$, up to first order terms in q_i, are as follows.

$$\begin{aligned} H_0 &= 1,\\ H_1 &= 2f_{20}q_1+f_{11}q_2,\\ H_2 &= -\beta_2+f_{11}\alpha_2q_1+2f_{02}\alpha_2q_2+f_{20}\widetilde{q_3},\\ H_3 &= -\beta_3+(-4\beta_2f_{20}+f_{11}\alpha_3)\,q_1+(-2\beta_2f_{11}+2f_{02}\alpha_3)\,q_2+f_{11}\alpha_2\widetilde{q_3}, \end{aligned}$$

Setting in these formulas $q_1=q_2=\widetilde{q_3}=0$ we get the following expressions of the Taylor coefficients of the organising centre $h_k=H_k|_{q_1=q_2=\widetilde{q_3}=0}$:

$$h_0 = 1,\quad h_1=0,\quad h_2=-\beta_2,\quad h_3=-\beta_3,\quad h_4=\beta_2^2-\beta_4$$

The function H has a singularity of type A_2 if $\beta_2=0$ and $\beta_3\neq 0$. If $\beta_2=\beta_3=0$ and $\beta_4\neq 0$ then the function H has a singularity of type A_3. More degenerate singularities are excluded due to genericity.

In order for H to be $\mathcal{R}^+$-versal with respect to q_1 and q_2 only at an A_k singularity for $k = 2, 3$ we need the first $k-1$ rows of the jet matrix

$$M_{k-1} = \begin{pmatrix} \frac{\partial^2 H}{\partial q_1 \partial t} & \frac{\partial^2 H}{\partial q_2 \partial t} \\ \\ \frac{\partial^3 H}{\partial q_1 \partial t^2} & \frac{\partial^3 H}{\partial q_2 \partial t^2} \end{pmatrix}$$

to have maximal rank $k-1$.

Versality of A_2 singularities on the surface M

An A_2 singularity occurs if $\beta_2 = 0$ and $\beta_3 \neq 0$. Recall that this is the necessary and sufficient condition that the tangent plane to the curve is the osculating plane with 3 point contact. The A_2 singularities are versal if the matrix

$$M_1 = \begin{pmatrix} 2f_{20} & f_{11} \end{pmatrix}$$

has rank 1.

Clearly the vanishing of β_2, f_{20} and f_{11} provide a set of non-generic conditions so A_2 singularities in the vicinity of the surface are versal.

Versality of A_3 singularities on the surface M

An A_3 singularity occurs if $\beta_2 = 0, \beta_3 = 0$ and $\beta_4 \neq 0$. This is the condition that the tangent plane to curve is the osculating plane and has 4 point contact (at a torsion zero). The A_3 singularities are versal if the matrix

$$M_2 = \begin{pmatrix} 2f_{20} & f_{11} \\ \alpha_2 f_{11} & 2\alpha_2 f_{02} \end{pmatrix}$$

has rank 2. The condition $\det(M_2) = 0$ together with the necessary conditions $\beta_2 = \beta_3 = 0$ singularity provide a non-generic condition so A_3 singularities are versal.

Since the generic singularities of the function H are $\mathcal{R}^+$-versal, lemma 3.4 implies that the generic singularities of the function $\mathcal{F}$ are space-time stable inside the space W.

At A_2 type points on the surface the caustic is smooth and transversally intersects the surface M. The respective generating family germ is space-time equivalent to the normal form:

$$\mathcal{F} = \lambda + \widetilde{q_3}(t^3 + q_1 t + 1)$$

At A_3 type points on the surface the caustic has a cuspidal edge that transversally intersects the surface M. The respective generating family germ is space-time equivalent to the normal form:

$$\mathcal{F} = \lambda + \widetilde{q_3}(t^4 + q_1 t^2 + q_2 t + 1). \quad \square$$

3.3. **Propagating from the surface in the Tangential Case in the vicinity of the surface.** In this case the caustic is space-time contact equivalent to one of the following normal forms (see [6]):

$$\widehat{B_2}: \quad \widetilde{q_3} + \lambda(t^2 + q_1); \qquad \widehat{B_3}: \quad \widetilde{q_3} + \lambda(t^2 \pm \lambda^2 + \lambda q_1 + q_2);$$

$$\widehat{C_3}: \quad \widetilde{q_3} + \lambda(t^3 + \lambda t + \lambda + q_1 t + q_2).$$

The caustic at a $\widehat{B_2}$ singularity consists only of the surface M and the criminant is a smooth surface with first order tangency with the surface M. At a $\widehat{B_3}$ singularity the criminant is diffeomorphic to a semi-cubic cylinder and has second order tangency with the surface M at a_0 (see figure 1). At a $\widehat{C_3}$ singularity the criminant is diffeomorphic to a folded Whitney umbrella and the caustic is a smooth surface. The cuspidal edge of the folded Whitney umbrella has first

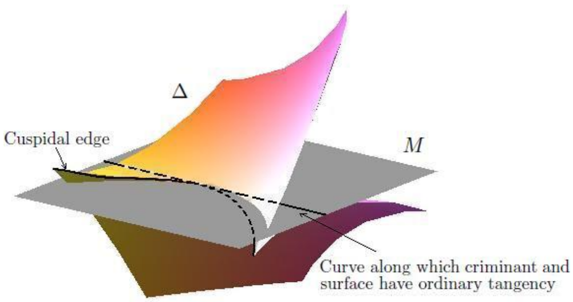

FIGURE 1. The envelope $B(M,N)$ at a $\widehat{B_3}$ singularity near the surface M (plane in figure). Here the caustic is empty and the criminant Δ is a cuspidal edge with second order tangency with the surface at a_0.

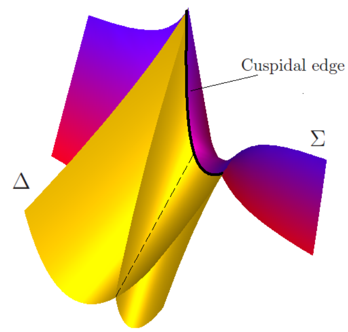

FIGURE 2. The caustic and the criminant shown together at a $\widehat{C_3}$ singularity. Here the criminant Δ is a folded Whitney umbrella and the caustic Σ is a smooth surface. The cuspidal edge of Δ has third order tangency with Σ.

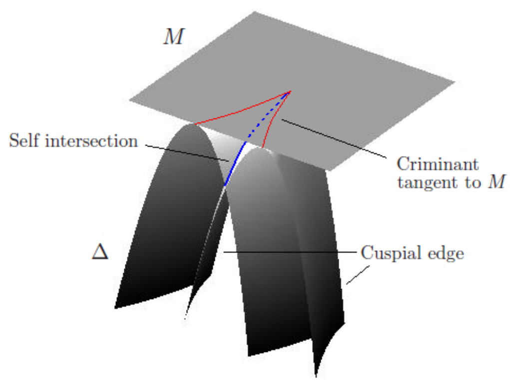

FIGURE 3. The criminant and the surface M (plane in figure) are shown together at a $\widehat{C_3}$ singularity. Here the criminant and M have ordinary tangency along a cusp.

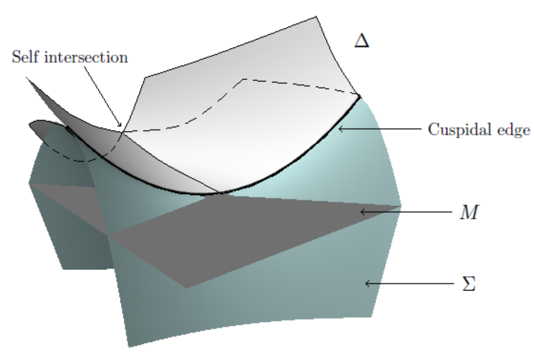

FIGURE 4. The envelope $B(M,N)$ at a $\widehat{C_3}$ singularity near the surface M (plane in figure). Here the criminant Δ is a folded Whitney umbrella and the caustic Σ is a smooth surface.

order tangency with the surface M at a_0 and third order tangency with the caustic (see figure 2). Two additional views are shown in figures 3 and 4.

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