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VARIETIES OF COMPLEXES AND FOLIATIONS

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Dedicated to Xavier Gómez-Mont on his 60th Birthday.

Abstract. Let $\mathcal{F}(r,d)$ denote the moduli space of algebraic foliations of codimension one and degree d in complex projective space of dimension r. We show that $\mathcal{F}(r,d)$ may be represented as a certain linear section of a variety of complexes. From this fact we obtain information on the irreducible components of $\mathcal{F}(r,d)$.

1. Basics on varieties of complexes.

1.1. Let K be a field and let $V_0,\dots,V_n$ be vector spaces over K of finite dimensions

$$d_i = \dim_K(V_i).$$

Consider sequences of linear functions

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} V_n \ ,$$

also written

$$f = (f_1,\dots,f_n) \in V = \prod_{i=1}^{n} \mathrm{Hom}_K(V_{i-1},V_i).$$

The variety of differential complexes is defined as

$$\mathcal{C} = \mathcal{C}(V_0,\dots,V_n) = \{f = (f_1,\dots,f_n) \in V/\ f_{i+1}\circ f_i = 0,\ i = 1,\dots,n-1\},$$

It is an affine variety in V, given as an intersection of quadrics. We intend to study the geometry of this variety (see also e.g., [3], [6]).

1.2. Since the defining equations $f_{i+1}\circ f_i = 0$ are bilinear, we may also consider, when it is convenient, the projective variety of complexes

$$P\mathcal{C} \subset \prod_{i=1}^{n} \mathbb{P}\mathrm{Hom}_K(V_{i-1},V_i),$$

as a subvariety of a product of projective spaces.

Denoting $V. = \oplus_{i=0}^{n} V_i$, each complex $f \in \mathcal{C}$ may be thought as a degree-one homomorphism of graded vector spaces $f : V. \to V.$ with $f^2 = 0$.

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1.3. For each $f \in \mathcal{C}$ and $i = 0, \dots, n$ define

$$B_i = f_i(V_{i-1}) \subset Z_i = \ker\,(f_{i+1}) \subset V_i,$$

and

$$H_i = Z_i / B_i.$$

(we understand by convention that $B_0 = 0$)

From the exact sequences

$$0 \to B_i \to Z_i \to H_i \to 0,$$

$$0 \to Z_i \to V_i \to B_{i+1} \to 0,$$

we obtain for the dimensions

$$b_i = \dim_K(B_i), \quad z_i = \dim_K(Z_i), \quad h_i = \dim_K(H_i),$$

the relations

$$d_i = b_{i+1} + z_i = b_{i+1} + b_i + h_i,$$

where $i = 0, \dots, n$ and $b_0 = b_{n+1} = 0$. Therefore,

Proposition 1. *a) The h_i and the b_j determine each other by the formulas:*

$$h_i = d_i - (b_{i+1} + b_i),$$

$$b_{j+1} = \chi_j(d) - \chi_j(h),$$

where for a sequence $e = (e_0, \dots, e_n)$ and $0 \leq j \leq n$ we denote

$$\chi_j(e) = (-1)^j \sum_{i=0}^{j} (-1)^i e_i = e_j - e_{j-1} + e_{j-2} + \cdots + (-1)^j e_0,$$

the j-th Euler characteristic of e.

b) The inequalities $b_{i+1} + b_i \leq d_i$ are satisfied for all i.

Proof. We write down the b_j in terms of the h_i: from

$$\sum_{i=0}^{j} (-1)^i d_i = \sum_{i=0}^{j} (-1)^i (b_{i+1} + b_i + h_i),$$

we obtain

$$b_{j+1} = (-1)^j \Big(\sum_{i=0}^{j} (-1)^i d_i - \sum_{i=0}^{j} (-1)^i h_i\Big),$$

as claimed.

□

Notice in particular that since $b_{n+1} = 0$, we have the usual relation

$$\sum_{i=0}^{n} (-1)^i d_i = \sum_{i=0}^{n} (-1)^i h_i.$$

1.4. Now we consider the subvarieties of $\mathcal{C}$ obtained by imposing rank conditions on the f_i.

Definition 2. *For each $r = (r_1, \dots, r_n) \in \mathbb{N}^n$ define*

$$\mathcal{C}_r = \{f = (f_1, \dots, f_n) \in \mathcal{C} /\ \operatorname{rank}(f_i) = r_i,\ i = 1, \dots, n\}.$$

These are locally closed subvarieties of $\mathcal{C}$.

Proposition 3. *a) $\mathcal{C}_r \neq \emptyset$ if and only if $r_{i+1} + r_i \leq d_i$ for $0 \leq i \leq n$ (we use the convention $r_0 = r_{n+1} = 0$)*

b) In the conditions of a), $\mathcal{C}_r$ is smooth and irreducible, of dimension

$$\dim(\mathcal{C}_r) = \sum_{i=0}^{n} (d_i - r_i)(r_{i+1} + r_i) = \sum_{i=0}^{n} (d_i - r_i)(d_i - h_i) = \frac{1}{2} \sum_{i=0}^{n} (d_i^2 - h_i^2).$$

Proof. a) One implication follows from Proposition 1. Conversely, in the given conditions, we want to construct a complex with $\operatorname{rank}(f_i) = r_i$ for all i. Suppose we constructed

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} V_{n-1}\ .$$

We need to define $f_n : V_{n-1} \to V_n$ such that $f_n \circ f_{n-1} = 0$ and $\operatorname{rank}(f_n) = r_n$, that is, a map $V_{n-1}/B_{n-1} \to V_n$ of rank r_n. Such a map exists since $\dim(V_{n-1}/B_{n-1}) = d_{n-1} - r_{n-1} \geq r_n$.

b) Consider the projection (forgeting f_n)

$$\pi : \mathcal{C}(V_0, \dots, V_n)_r \to \mathcal{C}(V_0, \dots, V_{n-1})_{\bar{r}},$$

where $r = (r_1, \dots, r_n)$ and $\bar{r} = (r_1, \dots, r_{n-1})$. Any fiber $\pi^{-1}(f_1, \dots, f_{n-1})$ is isomorphic to the subvariety in $\operatorname{Hom}(V_{n-1}/B_{n-1}, V_n)$ of maps of rank r_n; therefore, it is smooth and irreducible of dimension $r_n(d_{n-1} - r_{n-1} + d_n - r_n)$ (see [1]). The assertion follows by induction on n. The various expressions for $\dim(\mathcal{C}_r)$ follow by direct calculations.

Another proof of a): Given r such that $r_{i+1} + r_i \leq d_i$, put $h_i = d_i - (r_{i+1} + r_i) \geq 0$ and $z_i = d_i - r_{i+1} = h_i + r_i$. Choose linear subspaces $B_i \subset Z_i \subset V_i$ with $\dim(B_i) = r_i$ and $\dim(Z_i) = z_i$. Since $\dim(V_{i-1}/Z_{i-1}) = \dim(B_i)$, choose an isomorphism $\sigma_i : V_{i-1}/Z_{i-1} \to B_i$ for each i. Composing with the natural projection $V_{i-1} \to V_{i-1}/Z_{i-1}$ we obtain linear maps $V_{i-1} \to B_i$ with kernel Z_{i-1} and rank r_i, as wanted.

□

Remark 4. *In terms of dimension of homology, the condition in Proposition 3 a) translates as follows. Given $h = (h_0, \dots, h_n) \in \mathbb{N}^{n+1}$, there exists a complex with dimension of homology equal to h if and only if $\chi_i(h) \leq \chi_i(d)$ for $i = 1, \dots, n-1$ and $\chi_n(h) = \chi_n(d)$.*

Remark 5. *The group $G = \prod_{i=0}^{n}\ GL(V_i, K)$ acts on $V = \prod_{i=1}^{n} \operatorname{Hom}_K(V_{i-1}, V_i)$ via*

$$(g_0, g_1, \dots, g_n) \cdot (f_1, f_2, \dots, f_n) = (g_0 f_1 g_1^{-1}, g_1 f_2 g_2^{-1}, \dots, g_{n-1} f_n g_n^{-1}).$$

This action clearly preserves the variety of complexes. It follows from the proof above that the action on each $\mathcal{C}_r$ is transitive. Hence, the non-empty $\mathcal{C}_r$ are the orbits of G acting on $\mathcal{C}(V_0, \dots, V_n)$.

Definition 6. *For $r, s \in \mathbb{N}^n$ we write $s \leq r$ if $s_i \leq r_i$ for $i = 1, \dots, n$.*

Corollary 7. *If $\mathcal{C}_r \neq \emptyset$ and $s \leq r$ then $\mathcal{C}_s \neq \emptyset$. Also,* $\dim(\mathcal{C}_s) > 0$ *if* $s \neq 0$.

Proof. The first assertion follows from Proposition 3 a), and the second from Proposition 3 b). □

Proposition 8. *With the notation above,*

$$\overline{\mathcal{C}}_r = \bigcup_{s \leq r} \mathcal{C}_s = \{f \in \mathcal{C} / \ \operatorname{rank}(f_i) \leq r_i, \ i = 1, \ldots, n\}.$$

Proof. Denote $X_r = \bigcup_{s \leq r} \mathcal{C}_s$. Since the second equality is clear, X_r is closed. It follows that $\overline{\mathcal{C}}_r \subset X_r$. To prove the equality, since $\mathcal{C}_r \subset X_r$ is open, it would be enough to show that X_r is irreducible. For this, consider $L = (L_1, \ldots, L_n)$ where $L_i \in \text{Grass }(r_i, V_i)$ and denote

$$X_L = \{f = (f_1, \ldots, f_n) \in \mathcal{C} / \ \text{ im } (f_i) \subset L_i \subset \text{ ker } (f_{i+1}), \ i = 1, \ldots, n\}.$$

Consider

$$\tilde{X}_r = \{(L, f) / \ f \in X_L\} \ \subset \ G \times \mathcal{C},$$

where $G = \prod_{i=0}^n \text{Grass }(r_i, V_i)$. The first projection $p_1 : \tilde{X}_r \to G$ has fibers

$$p_1^{-1}(L) = X_L \cong \operatorname{Hom}(V_0, L_1) \times \operatorname{Hom}(V_1/L_1, L_2) \times \cdots \times \operatorname{Hom}(V_{n-1}/L_{n-1}, V_n),$$

which are vector spaces of constant dimension $\sum_{i=0}^n (d_i - r_i) r_{i+1}$. It follows that $\tilde{X}_r$ is irreducible, and hence $X_r = p_2(\tilde{X}_r)$ is also irreducible, as wanted.

□

Remark 9. *In the proof above we find again the formula*

$$\dim(X_r) = \dim(X_L) + \dim(G) = \sum_{i=0}^n (d_i - r_i) r_i + \sum_{i=0}^n (d_i - r_i) r_{i+1}.$$

Remark 10. *The fact that $p_1 : \tilde{X}_r \to G$ is a vector bundle implies that $\tilde{X}_r$ is smooth. On the other hand, since $p_2 : \tilde{X}_r \to X_r$ is birational (an isomorphism over the open set $\mathcal{C}_r$), it is a resolution of singularities.*

The following two corollaries are immediate consequences of Proposition 8.

Corollary 11. *$\mathcal{C}_s \subset \overline{\mathcal{C}}_r$ if and only if $s \leq r$.*

Corollary 12. *$\overline{\mathcal{C}}_r \cap \overline{\mathcal{C}}_s = \overline{\mathcal{C}}_t$ where $t_i = min\ (r_i, s_i)$ for all $i = 1, \ldots, n$.*

Definition 13. *For $d = (d_0, \ldots, d_n) \in \mathbb{N}^{n+1}$ let*

$$R = R(d) = \{(r_1, \ldots, r_n) \in \mathbb{N}^n / \ r_1 \leq d_0, \ r_{i+1} + r_i \leq d_i \ (1 \leq i \leq n-1), \ r_n \leq d_n\}.$$

We consider $\mathbb{N}^n$ ordered via $r \leq s$ if $r_i \leq s_i$ for all i; the finite set R has the induced order. Notice that R is finite since it is contained in the box $\{(r_1, \ldots, r_n) \in \mathbb{N}^n / \ 0 \leq r_i \leq d_i, \ i = 1, \ldots, n\}$.

Proposition 14. *With the notation above, the irreducible components of the variety of complexes $\mathcal{C} = \mathcal{C}(V_0, \ldots, V_n)$ are the $\overline{\mathcal{C}}_r$ with $r \in R(d_0, \ldots, d_n)$ a maximal element.*

Proof. From the previous Propositions, we have the equalities

$$\mathcal{C} = \bigcup_{r \in R} \mathcal{C}_r = \bigcup_{r \in R} \overline{\mathcal{C}}_r = \bigcup_{r \in R^+} \overline{\mathcal{C}}_r,$$

where R^+ denotes the set of maximal elements of R. The result follows because we know that each $\overline{\mathcal{C}}_r$ is irreducible and there are no inclusion relations among the $\overline{\mathcal{C}}_r$ for $r \in R^+$ (see Corollary 11).

□

1.5. **Morphisms of complexes. Tangent space of the variety of complexes.** Now we would like to compute the dimension of the tangent space of a variety of complexes at each point.

With the notation of 1.1 we consider complexes $f \in \mathcal{C}(V_0, \ldots, V_n)$ and $f' \in \mathcal{C}(V'_0, \ldots, V'_n)$ (the vector spaces V_i and V'_i are not necessarily the same, but the lenght n we may assume is the same). We denote

$$\mathrm{Hom}_{\mathcal{C}}(f, f'),$$

the set of morphisms of complexes from f to f', that is, collections of linear maps $g_i : V_i \to V'_i$ for $i = 0, \ldots, n$, such that $g_i \circ f_i = f'_i \circ g_{i-1}$ for $i = 1, \ldots, n$. It is a vector subspace of $\prod_{i=0}^n \mathrm{Hom}_K(V_i, V'_i)$, and we would like to calculate its dimension.

For this particular purpose and for its independent interest, we recall the following from [2] (§$2-5$. Complexes scindés):

For $f \in \mathcal{C}(V_0, \ldots, V_n)$, denote as in 1.1

$$B_i(f) = f_i(V_{i-1}) \subset Z_i(f) = \ker\,(f_{i+1}) \subset V_i.$$

Since we are working with vector spaces, we may choose linear subspaces $\bar{B}_i$ and $\bar{H}_i$ of V_i such that

$$V_i = Z_i(f) \oplus \bar{B}_i \quad \text{and} \quad Z_i(f) = B_i(f) \oplus \bar{H}_i.$$

Then $V_i = B_i(f) \oplus \bar{H}_i \oplus \bar{B}_i$ and clearly f_{i+1} takes $\bar{B}_i$ isomorphically onto $B_{i+1}(f)$. Notice also that

$$\dim(\bar{B}_i) = \dim(B_{i+1}(f)) = \mathrm{rank}(f_{i+1}) = r_{i+1}(f),$$

and

$$\dim(\bar{H}_i) = \dim(Z_i(f)/B_i(f)) = h_i(f).$$

Next, define the following complexes:

$\bar{H}(i)$ the complex of lenght zero consisting of the vector space $\bar{H}_i$ in degree i, the vector space zero in degrees $\neq i$, and all differentials equal to zero.

$\bar{B}(i)$ the complex of lenght one consisting of the vector space $\bar{B}_{i-1}$ in degree $i-1$, the vector space $B_i(f)$ in degree i, with the map $f_i : \bar{B}_{i-1} \to B_i(f)$, and zeroes everywhere else.

Proposition 15. *With the notation just introduced, $\bar{H}(i)$ and $\bar{B}(i)$ are subcomplexes of f and we have a direct sum decomposition of complexes:*

$$f = \bigoplus_{0 \le i \le n} \bar{H}(i) \;\oplus\; \bigoplus_{0 \le i \le n} \bar{B}(i).$$

Proof. Clear from the discussion above; see also [2], loc. cit. □

Now we are ready for the calculation of $\dim_K \mathrm{Hom}_{\mathcal{C}}(f, f')$.

Proposition 16. *With the previous notation, we have:*

$$\begin{aligned}\dim_K \operatorname{Hom}_{\mathcal{C}}(f,f') &= \sum_i h_i h'_i + h_i r'_i + r_i h'_{i-1} + r_i r'_i + r_i r'_{i-1} \\ &= \sum_i h_i(h'_i + r'_i) + r_i d'_{i-1}\end{aligned}$$

Proof. We may decompose f and f' as in Proposition 15:

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(f,f') &= \operatorname{Hom}_{\mathcal{C}}(\oplus_i \bar{H}(i) \oplus \oplus_i \bar{B}(i), \oplus_i \bar{H}(i)' \oplus \oplus_i \bar{B}(i)') \\ &= \oplus_{i,j}\operatorname{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{H}(j)') \ \oplus\ \oplus_{i,j}\operatorname{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{B}(j)') \oplus \\ &\quad \oplus_{i,j}\operatorname{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{H}(j)') \ \oplus\ \oplus_{i,j}\operatorname{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{B}(j)')\end{aligned}$$

It is easy to check the following:

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{H}(j)') &= 0 \ \text{ for } i \neq j \\ \operatorname{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{H}(i)') &= \operatorname{Hom}_K(\bar{H}_i, \bar{H}'_i)\end{aligned}$$

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{B}(j)') &= 0 \ \text{ for } i \neq j \\ \operatorname{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{B}(i)') &= \operatorname{Hom}_K(\bar{H}_i, \bar{B}'_i)\end{aligned}$$

(the case $j = i+1$ requires special attention)

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{H}(j)') &= 0 \ \text{ for } i-1 \neq j \\ \operatorname{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{H}(i-1)') &= \operatorname{Hom}_K(\bar{B}_{i-1}, \bar{H}'_{i-1}) \cong \operatorname{Hom}_K(\bar{B}_i(f), \bar{H}'_{i-1})\end{aligned}$$

(the case $j = i$ requires special attention)

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{B}(i)') &\cong \operatorname{Hom}_K(B_i(f), B'_i(f)) \\ \operatorname{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{B}(i-1)') &= \operatorname{Hom}_K(\bar{B}_{i-1}, B'_{i-1}) \cong \operatorname{Hom}_K(B_i(f), B'_{i-1}) \\ \operatorname{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{B}(j)') &= 0 \ \text{ otherwise}\end{aligned}$$

Taking dimensions we obtain the stated formula.

□

Now we deduce the dimension of the tangent space to a variety of complexes at any point.

Proposition 17. *For $f \in \mathcal{C} = \mathcal{C}(V_0, \ldots, V_n)$ we have a canonical isomorphism*

$$T\mathcal{C}(f) = \operatorname{Hom}_{\mathcal{C}}(f, f(1)),$$

where $T\mathcal{C}(f)$ is the Zariski tangent space to $\mathcal{C}$ at the point f, and $f(1)$ denotes de shifted complex $f(1)_i = (-1)^i f_{i+1}, \quad i = -1, 0, \ldots, n.$

Proof. Since $\mathcal{C}$ is an algebraic subvariety of the vector space $V = \prod_{i=1}^n \operatorname{Hom}_K(V_{i-1}, V_i)$, an element of $T\mathcal{C}(f)$ is a $g = (g_1, \ldots, g_n) \in V$ such that $f + \epsilon g$ satisfies the equations defining $\mathcal{C}$ (i.e., a $K[\epsilon]$-valued point of $\mathcal{C}$), that is,

$$(f+\epsilon g)_{i+1} \circ (f + \epsilon g)_i = 0, \quad i = 1, \ldots, n-1 \ \ (\text{modulo } \epsilon^2),$$

which is equivalent to

$$f_{i+1} \circ g_i + g_{i+1} \circ f_i = 0, \quad i = 1, \dots, n-1,$$

and this means precisely that $g \in \mathrm{Hom}_{\mathcal{C}}(f, f(1))$. $\square$

Corollary 18. *For $f \in \mathcal{C} = \mathcal{C}(V_0, \dots, V_n)$,*

$$\begin{aligned} \dim_K T\mathcal{C}(f) &= \sum_i h_i(h_{i+1} + r_{i+1}) + r_i d_i \\ &= \sum_i (d_i - r_i - r_{i+1})(d_{i+1} - r_{i+2}) + r_i d_i \end{aligned}$$

Proof. From Proposition 17 we know that $\dim_K T\mathcal{C}(f) = \dim_K \mathrm{Hom}_{\mathcal{C}}(f, f(1))$. Next we apply Proposition 16 with $f' = f(1)$, that is, replacing $d'_i = d_{i+1}$, $r'_i = r_{i+1}$, $h'_i = h_{i+1}$, to obtain the result.

$\square$

1.6. **Varieties of exact complexes.** Now we apply the previous results to the case of exact complexes.

Let us fix $(d_0, \dots, d_n) \in \mathbb{N}^n$ so that

$$\chi_j(d) = (-1)^j \sum_{i=0}^{j} (-1)^i d_i \geq 0, \quad j = 1, \dots, n-1,$$

$$\chi_n(d) = (-1)^n \sum_{i=0}^{n} (-1)^i d_i = 0.$$

Denoting $\chi = \chi(d) = (\chi_1(d), \dots, \chi_n(d)) \in \mathbb{N}^n$, let us consider the variety $\mathcal{C}_\chi$ of complexes of rank χ as in Definition 2 . Since $\chi_i(d) + \chi_{i-1}(d) = d_i$ for all i, it follows from Proposition 3 that $\mathcal{C}_\chi$ is non-empty of dimension

$$\frac{1}{2} \sum_{i=0}^{n} d_i^2.$$

It follows from Proposition 1 that any complex $f \in \mathcal{C}_\chi$ is exact. Also, since $\chi \in R$ is clearly maximal, $\overline{\mathcal{C}}_\chi$ is an irreducible component of $\mathcal{C}$ (see Proposition 14). Let us denote

$$\mathcal{E} = \mathcal{E}(d_0, \dots, d_n) = \overline{\mathcal{C}}_\chi = \{f \in \mathcal{C} /\ \mathrm{rank}(f_i) \leq \chi_i, \quad i = 1, \dots, n\},$$

the closure of the variety $\mathcal{C}_\chi$ of exact complexes. Denote also, for $i = 1, \dots, n$

$$\chi^i = \chi - e_i = (\chi_1, \dots, \chi_{i-1}, \chi_i - 1, \chi_{i+1}, \dots, \chi_n),$$

and

$$\Delta_i = \overline{\mathcal{C}}_{\chi^i} = \{f \in \mathcal{C} /\ \mathrm{rank}(f) \leq \chi - e_i\},$$

the variety of complexes where the i-th matrix drops rank by one.

Proposition 19. *The codimension of Δ_i in $\mathcal{E}$ is equal to one, and*

$$\mathcal{E} = \mathcal{C}_\chi \cup \Delta_1 \cup \dots \cup \Delta_n.$$

Proof. This follows from Proposition 8 and the fact that $s \in \mathbb{N}^n$ satisfies $s < \chi$ if and only if $s \leq \chi - e_i$ for some $i = 1, \dots, n$. $\square$

2. Moduli space of foliations.

2.1. Let X denote a (smooth, complete) algebraic variety over the complex numbers, let L be a line bundle on X and let ω denote a global section of $\Omega^1_X \otimes L$ (a twisted differential 1-form). A simple local calculation shows that $\omega \wedge d\omega$ is a section of $\Omega^3_X \otimes L^{\otimes 2}$. We say that ω is integrable if it satisfies the Frobenius condition $\omega \wedge d\omega = 0$. We denote

$$\mathcal{F}(X, L) \subset \mathbb{P}H^0(X, \Omega^1_X \otimes L),$$

the projective classes of integrable 1-forms. The map

$$\varphi : H^0(X, \Omega^1_X \otimes L) \to H^0(X, \Omega^3_X \otimes L^{\otimes 2}),$$

such that $\varphi(\omega) = \omega \wedge d\omega$ is a homogeneous quadratic map between vector spaces and hence $\varphi^{-1}(0) = \mathcal{F}(X, L)$ is an algebraic variety defined by homogeneous quadratic equations.

Our purpose is to understand the geometry of $\mathcal{F}(X, L)$. In particular, we are interested in the problem of describing its irreducible components. For a survey on this problem see for example [7].

2.2. Let r and d be natural numbers. Consider a differential 1-form in $\mathbb{C}^{r+1}$

$$\omega = \sum_{i=0}^{r} a_i dx_i,$$

where the a_i are homogeneous polynomials of degree $d-1$ in variables $x_0, \ldots, x_r$, with complex coefficients. We say that ω has degree d (in particular the 1-forms dx_i have degree one). Denoting R the radial vector field, let us assume that

$$< \omega, R >= \sum_{i=0}^{r} a_i x_i = 0,$$

so that ω descends to the complex projective space $\mathbb{P}^r$ as a global section of the twisted sheaf of 1-forms $\Omega^1_{\mathbb{P}^r}(d)$. We denote

$$\mathcal{F}(r, d) = \mathcal{F}(\mathbb{P}^r, \mathcal{O}(d)),$$

parametrizing 1-forms of degree d on $\mathbb{P}^r$ that satisfy the Frobenius integrability condition.

3. Complexes associated to an integrable form.

Let us denote

$$H^0(\mathbb{P}^r, \Omega^k_{\mathbb{P}^r}(d)) = \Omega^k_r(d),$$

and

$$\Omega_r = \bigoplus_{d\in\mathbb{N}} \bigoplus_{0\le k\le r} \Omega^k_r(d),$$

with structure of bi-graded supercommutative associative algebra given by exterior product $\wedge$ of differential forms.

Definition 20. *Gelfand, Kapranov and Zelevinsky defined in [5] another product in Ω_r, the second multiplication $*$, as follows:*

$$\begin{aligned}\omega_1 * \omega_2 &= \frac{d_1}{d_1+d_2}\omega_1\wedge d\omega_2 + (-1)^{(k_1+1)(k_2+1)}\frac{d_2}{d_1+d_2}\omega_2\wedge d\omega_1,\\ &= \frac{d_1}{d_1+d_2}\omega_1\wedge d\omega_2 + (-1)^{(k_1+1)}\frac{d_2}{d_1+d_2}d\omega_1\wedge \omega_2,\end{aligned}$$

where $\omega_i \in \Omega^{k_i}_r(d_i)$ for $i=1,2$ and $d_1+d_2\neq 0$. In case $(d_1,d_2)=(0,0)$ one defines $\omega_1*\omega_2=0$.

It follows that $\omega_1 * \omega_2 = 0$ if $d_1 = 0$ or $d_2 = 0$.

Remark 21. *For $\omega_i \in \Omega^{k_i}_r(d_i)$ for $i=1,2$ as above,*

a) $\omega_1*\omega_2$ belongs to $\Omega^{(k_1+k_2+1)}_r(d_1+d_2)$.

b) $\omega_1*\omega_2 = (-1)^{(k_1+1)(k_2+1)}\omega_2*\omega_1$.

c) It follows from an easy direct calculation that $*$ is associative (see [5]).

d) For any $\omega\in\Omega^1_r(d)$ we have $\omega*\omega = \omega\wedge d\omega$. In particular, ω is integrable if and only if $\omega*\omega=0$.

Definition 22. *For $\omega\in\Omega^k_r(d)$ we consider the operator δ_ω*

$$\delta_\omega : \Omega_r \to \Omega_r,$$

such that $\delta_\omega(\eta) = \omega*\eta$ for $\eta\in\Omega_r$.

Remark 23. *From Remark 21 a), if $\omega\in\Omega^{k_1}_r(d_1)$ then*

$$\delta_\omega(\Omega^{k_2}_r(d_2)) \subset \Omega^{(k_1+k_2+1)}_r(d_1+d_2).$$

In particular, if $\omega\in\Omega^1_r(d_1)$,

$$\delta_\omega(\Omega^{k_2}_r(d_2)) \subset \Omega^{(k_2+2)}_r(d_1+d_2).$$

Definition 24. *For $\omega\in\Omega^1_r(d)$ and $e\in\mathbb{Z}$ we define two differential graded vector spaces*

$$C^+_\omega(e) : \Omega^0_r(e) \to \Omega^2_r(e+d) \to \Omega^4_r(e+2d) \to \cdots \to \Omega^{2k}_r(e+kd) \to \dots,$$

$$C^-_\omega(e) : \Omega^1_r(e) \to \Omega^3_r(e+d) \to \Omega^5_r(e+2d) \to \cdots \to \Omega^{2k+1}_r(e+kd) \to \dots,$$

where all maps are δ_ω as in Remark 23.

Proposition 25. *Let $\omega \in \Omega^1_r(d)$, $e \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $k+2 \leq r$. Then $\omega * \eta = 0$ for all $\eta \in \Omega^k_r(e)$ if and only if $\omega = 0$. In other words, the linear map*

$$\delta : \Omega^1_r(d) \to \mathrm{Hom}_K(\Omega^k_r(e), \Omega^{k+2}_r(e+d)),$$

sending $\omega \mapsto \delta_\omega$, is injective.

Proof. First remark that $\omega \wedge \eta = 0$ for all $\eta \in \Omega^k_r(e)$ (with $k+1 \leq r$) easily implies $\omega = 0$. Now suppose $\omega * \eta = 0$, that is, $d\,\omega \wedge d\eta + e\,\eta \wedge d\omega = 0$, for all $\eta \in \Omega^k_r(e)$. Take

$$\eta = x_{i_1}^{e-k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

(here x_i denote affine coordinates and $1 < i_1 < \ldots i_k < n$). Since $d\eta = 0$, we have

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge d\omega = 0.$$

Hence $d\omega = 0$ by the first remark. Using the hypothesis again, we know $\omega \wedge d\eta = 0$ for all $\eta \in \Omega^k_r(e)$. Now take $\eta = x_{i_{k+1}}^{e-k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ (where $1 < i_1 < \cdots < i_{k+1} < n$). It follows that $dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} \wedge \omega = 0$ and hence $\omega = 0$. □

Proposition 26. *$\omega \in \Omega^1_r(d)$ is integrable if and only if $\delta^2_\omega = 0$*

Proof. The associativity stated in Remark 21 c) implies that $\delta_{\omega_1} \circ \delta_{\omega_2} = \delta_{\omega_1 * \omega_2}$. In particular, $\delta^2_\omega = \delta_{\omega * \omega}$ and hence the claim follows from Remark 21 d) and Proposition 25. □

Remark 27. *It follows from Proposition 26 that $C^+_\omega(e)$ and $C^-_\omega(e)$ (Definition 24) are differential complexes (for any $e \in \mathbb{Z}$) if and only if ω is integrable.*

Remark 28. *To fix ideas we shall mostly discuss $C^-_\omega(e)$, but similar considerations apply to $C^+_\omega(e)$. If no confusion seems to arise we shall denote $C^-_\omega(e) = C_\omega(e)$.*

Theorem 29. *Fix $e \in \mathbb{Z}$. Let us consider the graded vector space*

$$\Omega_r(e) = \bigoplus_{0 \leq k \leq [\frac{r-1}{2}]} \Omega^{2k+1}_r(e+kd),$$

(direct sum of the spaces appearing in $C^-_\omega(e)$ above). Define the linear map

$$\delta(e) = \delta : \Omega^1_r(d) \to \prod_{k=1}^{[\frac{r-1}{2}]} \mathrm{Hom}_K(\Omega^{2k-1}_r(e+(k-1)d), \Omega^{2k+1}_r(e+kd)),$$

such that $\delta(\omega) = \delta_\omega$ for each $\omega \in \Omega^1_r(d)$, and its projectivization

$$\mathbb{P}\delta : \mathbb{P}\Omega^1_r(d) \to \prod_{k=1}^{[\frac{r-1}{2}]} \mathbb{P}\mathrm{Hom}_K(\Omega^{2k-1}_r(e+(k-1)d), \Omega^{2k+1}_r(e+kd)).$$

Denote $\mathcal{C} = \mathcal{C}(\Omega^1_r(e), \Omega^3_r(e+d), \Omega^5_r(e+2d), \ldots, \Omega_r^{2[\frac{r-1}{2}]+1}(e+[\frac{r-1}{2}]d))$ the variety of complexes as in 1.1 and $\mathcal{F}(r,d)$ the variety of foliations as in 2.2. Then

$$\mathcal{F}(r,d) = (\mathbb{P}\delta)^{-1}(\mathcal{C}).$$

In other terms, $\mathbb{P}\delta(\mathcal{F}(r,d)) = L \cap \mathcal{C}$, that is, the variety of foliations $\mathcal{F}(r,d)$ corresponds via the linear injective map $\mathbb{P}\delta$ to the intersection of the variety of complexes with the linear space $L = \mathrm{im}(\mathbb{P}\delta)$.

Proof. The statement is a rephrasing of Remark 27. □

Proposition 30. *Let us denote*

$$d_r^k(e) = \dim \Omega_r^k(e) = \binom{r-k+e}{r-k}\binom{d-1}{k},$$

(see [8]) and in particular

$$d_k = d_r^{2k+1}(e+kd) = \dim \Omega_r^{2k+1}(e+kd),\ 0 \le k \le [\frac{r-1}{2}].$$

For this $d = (d_0, d_1, \dots, d_{[\frac{r-1}{2}]})$ we consider the finite ordered set $R = R(d)$ as in Proposition 14. Then each irreducible component of the variety of foliations $\mathcal{F}(r,d)$ is an irreducible component of the linear section $(\mathbb{P}\delta)^{-1}(\overline{\mathcal{C}}_r)$ for a unique $r \in R^+$.

Proof. From Proposition 14, we have the decomposition into irreducible components

$$\mathcal{C} = \bigcup_{r \in R^+} \overline{\mathcal{C}}_r.$$

From Theorem 29 we obtain:

$$\mathcal{F}(r,d) = (\mathbb{P}\delta)^{-1}(\mathcal{C}) = \bigcup_{r \in R^+} (\mathbb{P}\delta)^{-1}(\overline{\mathcal{C}}_r).$$

This implies that each irreducible component X of $\mathcal{F}(r,d)$ is an irreducible component of $(\mathbb{P}\delta)^{-1}(\overline{\mathcal{C}}_r)$ for some $r \in R^+$. This element r is the sequence of ranks of δ_ω for a general $\omega \in X$, hence it is unique. □

References

[1] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris *Geometry of Algebraic Curves, vol. 1*. Springer-Verlag, 1984.

[2] N. Bourbaki *Algebre, Chapitre 10, Algebre Homologique*. Masson, 1980.

[3] M. Brion *Groupe de Picard et nombres caracteristiques des varietes spheriques*. Duke Math. J., Vol. **58**, (1989), 397–425.

[4] C. Camacho and A. Lins Neto, *The topology of integrable differential forms near a singularity.* Inst. Hautes Études Sci. Publ. Math. No. **55**, (1982), 5–35.

[5] I. Gelfand, M. Kapranov and A. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*. Birkhauser, 1994. DOI: 10.1007/978-0-8176-4771-1

[6] G. Kempf, *Images of homogeneous vector bundles and varieties of complexes*. Bulletin of the AMS **81**, (1975).

[7] A. Lins Neto, *Componentes irredutiveis dos espaos de folheacoes*. Rio de Janeiro, 2007.

[8] C. Okonek, M. Schneider and H. Spindler, *Vector Bundles on Complex Projective Spaces*. Birkhauser, 1980.

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