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A HOMOTOPY INVARIANT OF IMAGE SIMPLE FOLD MAPS TO ORIENTED SURFACES

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Abstract. The singular set of a generic map from a closed manifold of dimension at least 2 to the plane is a smooth closed curve. We study the parity of the number of components of the singular set under the assumption that the map is an image simple fold map, i.e., the map's restriction to its singular set is a smooth embedding.

The image of the singular set of a map to a plane inherits canonical local orientations via so-called chessboard functions. Such a local orientation gives rise to a cumulative winding number, which is an integer or a half integer. When the dimension of the source manifold is even, we also define an invariant I which is the residue class modulo 4 of the sum of twice the number of components of the singular set, the number of cusps, and twice the number of self-intersection points of the image of the singular set. Using the cumulative winding number and the invariant I, we show that the parity of the number of connected components of the singular set does not change under homotopy between image simple fold maps provided that one of the following conditions is satisfied: (i) the dimension of the source manifold is even, (ii) the image of the singular set of the homotopy does not have triple self-intersection points, or (iii) the singular set of the homotopy is an orientable manifold with boundary.

1. Introduction

Singular sets of smooth maps $f : M \to F$ of smooth n-manifolds into surfaces played a strong role in recent various discoveries. Studying singular sets of maps, Gay and Kirby [5] proved that any smooth closed oriented connected 4-manifold admits a trisecting map to $\mathbb{R}^2$, in analogy to the existence of Heegaard splittings for oriented connected closed 3-manifolds, see also the paper [3] by Baykur and Saeki for the existence of a simplified trisection. Kalmar and Stipsicz [9] obtained upper bounds on the complexity of the singular set of maps from 3-manifolds to the plane. These upper bounds are expressed in terms of certain properties of the link $L \subset S^3$, where the 3-manifold is obtained via integral surgery along L. Ryabichev [16] gave precise conditions for the existence of maps of surfaces with prescribed loci of singularities. Kitazawa [10] studied simple stable maps (of non-negative dimension) of smooth manifolds to Euclidean target spaces, ($\mathbb{R}^2$, in particular) whose singular sets are concentric spheres. Saeki [18] and [19] showed that every closed connected oriented 3-manifold admits a stable map to a sphere without definite fold points. Many $\mathbb{Z}_2$-invariants of stable maps of 3-manifolds into the plane were found by M. Yamamoto in [22]. In [20] Saeki constructed an integral invariant of stable maps of oriented closed 3-manifolds into $\mathbb{R}^2$.

A generic smooth map is *image simple* if its restriction to the singular set is a topological embedding. In the present paper we study under what conditions the numbers $\#|\Sigma(f)|$ and $\#|\Sigma(g)|$ of components of singular sets of two homotopic image simple fold maps f and g of manifolds of dimension $m \geqslant 2$ to a surface are congruent modulo two.

This question has been solved in the case $m = 2$, and it is partially answered in the case $m = 3$. Namely, M. Yamamoto [23] showed that if f is a map of degree d between oriented closed surfaces of genera g and h respectively, then the parity of $\#|\Sigma(f)|$ is the same as that of $d(h-1)-(g-1)$. On the other hand, in [20] Saeki studied maps of 3-manifolds into surfaces,

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and, in particular, gave an example of two image simple fold maps $f, g\colon S^3 \to \mathbb{R}^2$ such that the parities of the number of components of the singular sets of f and g are different.

Our main result is split into three cases; the first being the case when the source manifold is of even dimension.

Theorem 1.1. *Let f and g be two homotopic image simple fold maps from a closed manifold M of even dimension $m \geqslant 2$ to an oriented surface F of finite genus. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.*

To prove Theorem 1.1 we define the cumulative winding number $\omega(f) \in \frac{1}{2}\mathbb{Z}$ for generic maps to parallelized surfaces. In general, ω is not a homotopy invariant. However, for image simple fold maps $f, g\colon M \to F$ to parallelized surfaces, the invariant ω is integral, and the parities of $\omega(f)$ and $\omega(g)$ agree. Thus, for image simple fold maps, $\omega \in \mathbb{Z}$ is a $\mathbb{Z}_2$-homotopy invariant. We note that the cumulative winding number we introduce in the present paper is different from the rotation numbers considered by Levine [12], Chess [4], and Yonebayashi [24].

We now state theorems for the remaining two cases; the first theorem requires that $\Sigma(f)$ does not undergo any R_3 moves (see Fig. 2) during homotopy, while the second requires that the singular set of the homotopy is orientable.

Theorem 1.2. *Let f and g be two homotopic image simple fold maps $M \to F$, where*

- *M is a closed manifold of odd dimension $m > 2$ and F is $\mathbb{R}^2$ or S^2, or*
- *M is a closed oriented manifold of dimension 3, and F is an oriented surface.*

Suppose that no R_3 moves occur during a generic homotopy from f to g. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.

We note that R_3-moves are closely related to triple points of the singular sets $\Sigma(h)$ of maps h to $\mathbb{R}^3$. These are studied by Saeki and T. Yamamoto [21].

The proof of Theorem 1.2 also utilizes the cumulative winding number $\omega(f)$.

Theorem 1.3. *Let f and g be two homotopic image simple fold maps from a closed manifold M of dimension $m \geqslant 2$ to a surface F of finite genus. Suppose the surface $\Sigma(H)$ of singular points of the homotopy H between f and g is orientable. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.*

Let $\#|A_2(f)|$ be the number of cusps of the map f, $\Delta(f)$ the number of self-intersection points of $f(\Sigma)$, and $\#|\Sigma(f)|$ the number of connected components of $f(\Sigma)$. To prove Theorem 1.3, we introduce a modulo 4 function

$$I(f) \equiv \#|A_2(f)| + 2\Delta(f) + 2\#|\Sigma(f)| \pmod 4,$$

and show that it is invariant under generic homotopy whose singular set is orientable. In particular, the function $I(f)$ is a homotopy invariant provided that the dimension of the manifold M is even and the surface F is orientable by Theorem 3.4. In [7], Gromov introduced and more deeply studied $I(f)$ as an integer-valued function.

The paper is structured as follows. In section 2 we review the notions of generic maps, stable maps, and generic families of maps. We note that there are several conflicting definitions of a generic family of maps in the literature and chose one which is the most convenient for the present paper. In section 3 we review singularities $A_i(f)$ of Morin maps and introduce the manifolds $A_I(f) \subset M$ related to multi-singularities of smooth maps. In section 4, using the manifolds $A_I(f)$, we list all moves of singularities which occur under a generic homotopy of maps to $\mathbb{R}^2$. For completeness, we give a proof that no other moves are possible. Section 5 serves to introduce the notion of an abstract singular set diagram. In section 6 we define chessboard functions and in section 7 we look at examples of chessboard functions. In sections 8 and 9 we define the cumulative winding number and record how homotopy affects the cumulative winding number, respectively. In section 10 we prove Theorems 1.1 and 1.2, and in section 11, we prove that $I(f)$

is indeed invariant under homotopy with orientable singular set, and use it to prove Theorem 1.3. We finish our discussion in section 12 by listing and proving a few applications of our results.

We would like to express our thanks to Osamu Saeki and Masamichi Takase for their comments and references.

2. Stable and generic maps

In this section we recall the definition of stable maps, generic maps, generic families of maps, and n-functions.

Let f be a smooth map of a non-negative dimension $m - n$ of a manifold M of dimension m to a manifold N of dimension n. We say that a point $x \in M$ is *regular* if the kernel rank of f at x is $m - n$. Otherwise, the point x is said to be *singular*. Recall that a smooth map is a *Thom-Boardman* map if for each k, its k-jet extension is transverse to each Thom-Boardman submanifold of the k-jet space. The singular set $\Sigma(f)$ of a Thom-Boardman map $f\colon M \to N$ is stratified by smooth submanifolds $\Sigma^I(f) \subset M$ parametrized by Thom-Boardman symbols I.

2.1. **Generic maps.** Let $f\colon M \to N$ be a Thom-Boardman map. Let $x_j \in \Sigma^{I_j}(f)$ be distinct singular points in M with $j = 1, ..., r$ such that

$$f(x_1) = f(x_2) = \cdots = f(x_r) = y.$$

We say that f satisfies the *normal crossing condition* if for each tuple of points $x_1, ..., x_r$ as above the vector spaces

$$d_{x_1} f(T_{x_1} \Sigma^{I_1}), ..., d_{x_r} f(T_{x_r} \Sigma^{I_r})$$

are in general position in the vector space $T_y N$.

Definition 2.1. We say that a smooth map f is *generic* if it is a Thom-Boardman map satisfying the normal crossing condition.

It is known that generic maps are residual in $C^\infty(M, N)$, e.g. see [6, p.157].

2.2. **Stable maps.** A smooth map $f\colon M \to N$ is said to be *stable* if any smooth map f' sufficiently close to f is *right-left equivalent* to f, i.e., there are diffeomorphisms g of N and h of M such that $f' = g \circ f \circ h^{-1}$. In fact, there are various equivalent definitions of stability of smooth maps $f\colon M \to N$ of a closed manifold M to an arbitrary manifold N, e.g., see [6, Chapter V, Theorem 7.1]. In particular, f is stable if and only if any k-parametric deformation of f is trivial in the sense of [6, Chapter V, Definition 2.3].

Stable maps are generic, e.g., see [6, Chapter VI, Theorem 5.2]. On the other hand, it is known that stable maps are not dense in $C^\infty(M, N)$, e.g., see [6, p. 160]. In particular, the sets of stable maps and generic maps are not the same in general. However, a proper map of a manifold of dimension m to a manifold of dimension $n \leqslant 3$ such that $m \geqslant n$ is stable if and only if it is generic [14].

2.3. **Generic families of maps.** Let $f_t\colon M \to N$ be a parametric family of maps parametrized by a smooth manifold T. It defines a map $F\colon M \times T \to N \times T$ by $F(x, t) = (f_t(x), t)$, and a stratification of $M \times T$ by submanifolds $\Sigma^I(F)$, where I ranges over Thom-Boardman symbols. It is common to define a *generic homotopy* f_t by requiring that the associated map F is generic. However, we will need a more restrictive definition. Let π_T denote the projection of $M \times T$ onto the second factor. We say that a parametric family $\{f_t\}$ is a *generic parametric family* if the associated map F is generic, and the restrictions $\pi_T|_{\Sigma^I(F)}$ are generic for each Thom-Boardman symbol I. A parametric family $\{f_t\}$ is a *stable parametric family* if any k-parametric deformation of $F(x, t) = (f_t(x), t)$ is trivial.

2.4. **n-functions.** In some cases it is helpful to study maps to manifolds of dimension n by means of $(n-1)$-parametric families of functions, or, n-functions. More precisely, given a manifold X of dimension m, and a manifold Y of dimension $n \leqslant m$, a smooth proper map $f\colon X \to Y$ is an *n-function* if for each $q \in Y$, there is a compact neighborhood U of q with a diffeomorphism $\psi\colon U \to [0,1]^n$, and a diffeomorphism $\varphi\colon f^{-1}(U) \to [0,1]^{n-1} \times M$ for an $(m-n+1)$-manifold M, such that $\psi \circ f \circ \varphi^{-1} : [0,1]^{n-1} \times M \to [0,1]^{n-1} \times [0,1]$ is of the form $(t,p) \mapsto (t, g_t(p))$, for some parametric family g_t of functions on M. A generic 2-function is also called a Morse 2-function, see [5, Definition 2.7].

Lemma 2.2. *Let $f\colon X \to Y$ be a generic smooth proper map of corank 1 to a manifold of dimension n. Then f is an n-function.*

Proof. Let q be a point in Y. Since f is of corank 1, there is a diffeomorphism $\psi\colon U \to [0,1]^n$ of a neighborhood U of q such that the composition $\pi_n^{\perp} \circ \psi \circ f|_{f^{-1}(U)}$ is a submersion, where $\pi_n^{\perp}\colon [0,1]^n \to [0,1]^{n-1}$ is the projection $(x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1})$. We may choose U so that the resulting proper submersion to a disc is a trivial fiber bundle. Then, there is a diffeomorphism $\varphi\colon f^{-1}(U) \to [0,1]^{n-1} \times M$ such that the map $\psi \circ f \circ \varphi^{-1}$ is of the form $(t,p) \to (t, g_t(p))$, for a parametric family g_t of functions on a manifold M of dimension $m-n+1$. ☐

3. Singularities of maps

In this section we review the definition of generic singularities of smooth maps to surfaces and manifolds of dimension 3.

Let f be a smooth map $f\colon M \to N$ of non-negative dimension $m-n$ of a manifold M of dimension m to a manifold N of dimension n. The set $A_0(f)$ of regular points of f is an open submanifold of M of codimension 0. We now review the definition of singularity types A_r for $r \geqslant 1$ with Thom-Boardman symbol $I_r = (m-n+1, 1, ..., 1, 0)$ of length $r+1$.

We say that a point $x \in M$ is a *fold point* if there is a neighborhood $U \cong \mathbb{R}^{n-1} \times \mathbb{R}^{m-n+1}$ about x, with coordinates $(x_1, ..., x_m)$ in M, and a coordinate neighborhood $V \cong \mathbb{R}^{n-1} \times \mathbb{R}$ about $f(x)$ in N such that $f(U) \subset V$ and $f|_U$ is given by a product of the identity map $\mathrm{id}_{\mathbb{R}^{n-1}} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ and a standard Morse function $\mathbb{R}^{m-n+1} \to \mathbb{R}$ with a unique critical point, i.e.,

$$f(x_1, x_2, ..., x_m) = (x_1, ..., x_{n-1}, \pm x_n^2 \pm x_{n+1}^2 \pm ... \pm x_m^2). \tag{1}$$

The set of fold points of f is denoted by $A_1(f)$. The number i of terms in (1) among $x_n, ..., x_m$ with negative signs is called a *relative index* of f. We may always choose coordinate neighborhoods so that $i \leqslant m-n+1-i$. The number i with respect to such a coordinate system is said to be the (absolute) *index* of the fold point. If the index of the critical point is 0, then x is said to be a *definite* fold point. Otherwise, the fold point x is *indefinite*.

Definition 3.1. We say that the map f is a *fold map* if every singular point x is a fold point. Furthermore, a fold map f is an *indefinite fold map* if every fold point is indefinite.

It immediately follows that if f is a fold map, then the set of singular points $\Sigma(f)$ of f is a closed submanifold of M of dimension $n-1$, and $f|_{\Sigma(f)}$ is an immersion.

We say that a point $x \in M$ is an A_r-singular point for $r > 1$, if there is a neighborhood $U \subset M$ of x, with coordinates $(t_1, ..., t_{n-r}, \ell_2, ..., \ell_r, x_1, ..., x_{m-n+1})$, and a neighborhood $V \subset N$ of $f(x)$, with coordinates $(T_1, ..., T_{n-r}, L_2, ..., L_r, Z)$, such that $f(U) \subset V$ and the restriction $f|_U$ is given by

$$\begin{gathered} T_i = t_i \qquad \text{for} \quad i = 1, ..., n-r, \\ L_i = \ell_i \qquad \text{for} \quad i = 2, ..., r, \\ Z = \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_{m-n}^2 + \ell_2 x_{m-n+1} + \ell_3 x_{m-n+1}^2 + \cdots + \ell_r x_{m-n+1}^{r-1} \pm x_{m-n+1}^{r+1}. \end{gathered}$$

Given a map f, the sets $A_r(f)$ of its singular points of type A_r are submanifolds of M of dimension $n-r$.

Definition 3.2. Singular points of types A_2 and A_3 are called *cusp* and *swallowtail* singular points, respectively.

Definition 3.3. We say that a stable map $f : M \to N$ of a smooth manifold M into a surface N is *simple* if $A_2(f) = \varnothing$, and for every singular value y, every connected component of the singular fiber $f^{-1}(y)$ contains at most one singular point. Also, a generic map $f : M \to N$ is said to be *image simple* if its restriction to the singular set $f|_{\Sigma(f)}$ is a topological embedding.

We note that the term 'image simple map' is introduced by Saeki in his forthcoming paper.

3.1. **Morin Maps.** We say that a smooth map f is a *Morin map* if all its singular points are of type A_r for $r \geqslant 1$. It is known that for $n \leqslant 3$, all generic maps $M^m \to \mathbb{R}^n$ of non-negative dimension $m-n$ are Morin. The singular set $\Sigma(f)$ of a Morin map is a closed smooth submanifold of M of dimension $n-1$. Given a Morin map f, for each i, the closure $\mathrm{Cl}(A_i(f))$ is a smooth submanifold of M possibly with boundary. Furthermore, for each i and j such that $i < j$, the manifold $\mathrm{Cl}(A_j(f))$ is a submanifold of $\mathrm{Cl}(A_i(f))$. For a generic Morin map f, we denote by $A_{ij}(f)$ the set of points $x \in A_i(f)$ for which there is a distinct point $y \in A_j(f)$ such that $f(x) = f(y)$. Similarly, we denote by $A_{ijk}(f)$ the subset of points $x \in A_i$ for which there are distinct points $y \in A_j(f)$ and $z \in A_k(f)$ such that $f(x) = f(y) = f(z)$. We will denote the restriction of f to $A_I(f)$ by $f|_{A_I}$, for short, where I is either a single index i, or a multi-index ij or ijk.

When the non-negative dimension $m-n$ of a map $f : M \to N$ is even, we have the following theorem.

Theorem 3.4. *Let $f : M \to N$ be a Morin map of non-negative even dimension $m-n$ into an oriented manifold N. Then, the set $\Sigma(f)$ is a canonically oriented submanifold of M.*

We emphasize that the manifold M in Theorem 3.4 is not necessarily orientable.

Proof. Since $\mathrm{Cl}(A_3(f))$ is a proper submanifold of $\Sigma(f)$ of codimension 2, the manifold $\Sigma(f)$ is orientable if and only if the manifold $\Sigma(f) \setminus \mathrm{Cl}(A_3(f))$ is orientable. Thus, to prove Theorem 3.4, it suffices to introduce an orientation of $\Sigma(f)$ in the complement to $\mathrm{Cl}(A_3(f))$, i.e., only over the union of $A_1(f)$ and $A_2(f)$.

We note that the index of a fold point x depends on the choice of coorientation of the immersed submanifold $f(A_1)$ at the point $f(x)$. If a fold point x is of index i for one choice of coorientation, then x is of index $m-i-n+1$ for the other choice of coorientation. Since the (non-negative) dimension $m-n$ of the map f is even, it follows that the parity of the index is changed when the coorientation is changed. Consequently, the immersed manifold of fold points $f(A_1)$ admits a unique coorientation at each point $f(x)$, such that the index of the fold point x with respect to the coorientation is odd. We say that such a coorientation is *canonical*.

We orient the immersed manifold $f(A_1)$ so that the orientation of $f(A_1)$ followed by the coorientation of $f(A_1)$ agrees with the orientation of N. In turn, the orientation of $f(A_1)$ defines an orientation of $A_1(f)$. We claim that the so-defined orientation of $A_1(f)$ extends to an orientation of $A_1(f) \cup A_2(f)$. Indeed, in a coordinate neighborhood U about an A_2-singular point x and a coordinate neighborhood about $f(x)$, the map f is given by:

$$T_i = t_i, \qquad i = 1, ..., n-2,$$

$$L_2 = \ell_2,$$

$$Z = \varphi_{\ell_2}(x_{m-n+1}) \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_{m-n}^2,$$

where

$$(t_1, .., t_{n-2}, l_2, x_1, ..., x_{m-n}, x_{m-n+1})$$

are local coordinates about x in M and

$$(T_1, T_2, ..., T_{n-2}, L_2, Z)$$

are local coordinates about $f(x)$ in N, and for each l_2,

$$\varphi_{\ell_2}(x_{m-n+1}) = \ell_2 x_{m-n+1} + x_{m-n+1}^3$$

is either a regular function, a Morse function with a cancelling pair of critical points, or a function with a unique birth-death singularity. For each fold critical value in $f(\Sigma \cap U)$, the direction $\frac{\partial}{\partial Z}$ defines a coorientation of $f(\Sigma)$ at the corresponding critical value. It follows that for each Morse function $\varphi_{(t_1,\dots,t_{n-2},\ell_2)}$, the parities of the indices of the two cancelling Morse critical points are different. Therefore, the canonical coorientation of one critical point of $\varphi_{(t_1,\dots,t_{n-2},\ell_2)}$ is given by $\frac{\partial}{\partial Z}$, while the canonical coorientation for the other critical point is $-\frac{\partial}{\partial Z}$. Thus, the coorientation of $f(A_1)$ extends to a coorientation of an immersed smoothing of $f(A_1 \cup A_2)$. This implies that $\Sigma(f)$ is orientable. □

3.2. **Singularities of generic maps to 2-manifolds.** Let $f\colon M \to N$ be a generic smooth map of a manifold of dimension $m \geqslant 2$ to a manifold N of dimension 2. The map f may only have regular, fold, and cusp points. The set of regular points forms an open submanifold $A_0(f)$ of M. The complement to the submanifold $A_0(f)$ in M is the submanifold of singular points $\Sigma(f)$ of dimension 1. It contains a discrete set of cusp singular points $A_2(f)$. The rest of $\Sigma(f)$ is a disjoint union of arcs and circles of fold points $A_1(f)$. The restriction of f to $A_0(f)$ is a submersion. The restriction of f to $A_1(f)$ is a self-transverse immersion with 0-dimensional self-crossings. The images of $f|_{A_1}$ and $f|_{A_2}$ are disjoint.

3.3. **Singularities of generic maps to 3-manifolds.** Let $F\colon M \to N$ be a generic smooth map of a manifold of dimension $m \geqslant 3$ to a manifold of dimension 3. The map F may only have regular, fold, cusp, and swallowtail map germs. Since F satisfies the normal crossing condition, the set $A_{11}(F)$ is a submanifold which consists of open arcs and circles. We note that the image of $A_{11}(F)$ is the self-crossing of the immersion $F|_{A_1}$, while the image of $A_{12}(F) \cong A_{21}(F)$ is the set of intersections of folds with cusps. The image of the set $A_{111}(F)$ is the set of triple self-intersections of folds. The submanifolds $A_{12}(F) \subset A_1(F)$, $A_{21}(F) \subset A_2(F)$ and $A_{111}(F)$ are of dimension 0, while all other manifolds $A_{ij}(F)$ and $A_{ijk}(F)$ (except for the aforementioned manifold $A_{11}(F)$) are empty.

4. Generic homotopies of maps to $\mathbb{R}^2$

In this section we study how the singular set of a map to $\mathbb{R}^2$ is modified under generic homotopy, i.e., under generic one parameter family of maps.

Let $F\colon M \times [0,1] \to \mathbb{R}^2 \times [0,1]$ be a homotopy between two generic maps of a closed manifold M, and let $\pi\colon M \times [0,1] \to [0,1]$ denote the projection onto the second factor. Then the homotopy F is a *generic homotopy* if F is a generic map and $\pi|_{A_I(F)}$ is a generic function for each $I \in \{\{1\},\{2\},\{11\}\}$, see §2.3.

Lemma 4.1. *The set of generic homotopies is open and dense in the space of all homotopies.*

Proof. Any homotopy F' sufficiently close to a generic homotopy F is also generic. Indeed, a map to a manifold of dimension 3 is generic if and only if it is stable. In particular, a generic homotopy F is a stable map, see §2.2. Hence any homotopy F' sufficiently close to F is right-left equivalent to F, and in particular, generic. Consequently, the set of generic homotopies is open.

Next, let us show that the set of generic homotopies is dense. By the Mather theorem [15], stable maps to manifolds of dimension 3 are dense. Consequently, any homotopy F can be approximated by a stable map $F'\colon (M; M \times \{0\}, M \times \{1\}) \to ([0,1], 0, 1)$. On the other hand, an approximation of a homotopy is a homotopy. Thus, every homotopy F can be approximated by a homotopy that is a stable map. In particular, we may assume that F is itself a generic map.

If F is a generic map, then $\overline{A_1(F)}$ is a closed surface, while $\overline{A_2(F)}$ and $\overline{A_{11}(F)}$ are closed curves. There is a diffeomorphism $\varphi \in C^\infty(M \times [0,1], M \times [0,1])$ close to the identity map of $M \times [0,1]$ such that the restrictions of $\pi\colon M \times [0,1] \to [0,1]$ to $A_I(F \circ \varphi)$ are generic functions for each $I \in \{\{1\},\{2\},\{1,1\}\}$. If φ is chosen sufficiently close to the identity map of $M \times [0,1]$, then $F \circ \varphi$ is a generic homotopy close to F. Thus, every neighborhood of F contains a generic homotopy, i.e., the set of generic homotopies is dense. This completes the proof of Lemma 4.1. $\square$

We note that members f_t of a generic family $F = \{f_t\}$ may not be generic maps. We will next list several instances when a member f_t of a generic homotopy of maps to $\mathbb{R}^2$ is not generic. This list is exhaustive when F is a generic homotopy, see Theorem 4.2.

4.1. **List of generic moves.**

4.1.1. *Reidemeister-II fold crossing.* The restriction $f_t|_{A_1}$ may not be a self-transverse immersion for a discrete set of moments $t \in [0,1]$. If f_t is a generic homotopy, and $f_t|_{A_1}$ is not self-transverse at $t = t_0$, then as t ranges in the interval $(t_0 - \varepsilon, t + \varepsilon)$, the map $f_t|_{A_1}$ undergoes a Reidemeister-II fold crossing, see Fig. 1.

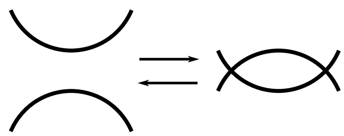

FIGURE 1. Reidemeister-II fold crossing

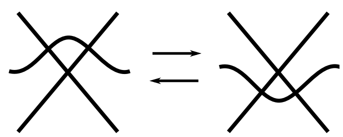

FIGURE 2. Reidemeister-III fold crossing

4.1.2. *Reidemeister-III fold crossing.* Similarly, the map $f_t|_{A_1}$ may undergo a Reidemeister-III fold crossing, see Fig. 2

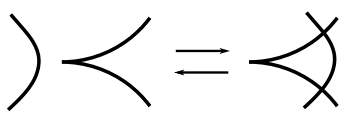

FIGURE 3. A cusp passing through a fold curve

4.1.3. *Cusp-fold crossing.* The cusp-fold crossing occurs when $f_t(x) = f_t(y)$, for a cusp point $x \in A_2(f_t)$ and a fold point $y \in A_1(f_t)$, see Fig. 3.

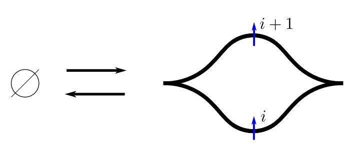

FIGURE 4. Wrinkle singularity

In Figures 4, 5, and 6, the numbers i and $i+1$ indicate the relative index of each fold curve. The relative index for each curve is considered in the direction of the corresponding blue arrow.

4.1.4. *Wrinkle singularity.* Under a generic homotopy, a new path component of singular points may appear in the form of a wrinkle, see Fig. 4.

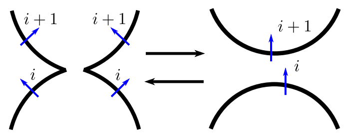

FIGURE 5. Merging and unmerging 2 cusps

4.1.5. *Merge singularity.* Under a merge singularity move, a canceling pair of cusp points disappear while the singular set changes by a surgery of index 1 along the canceling pair of cusp points, see Fig. 5.

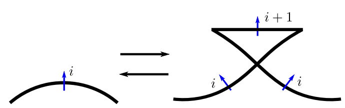

FIGURE 6. Introduction of a swallowtail

4.1.6. *Swallowtail singularity.* Under a swallowtail singularity move, two cusp points and a self-intersection point of the singular set appear, see Fig. 6.

Theorem 4.2. *Under a generic homotopy $F = \{f_t\}$ of maps to $\mathbb{R}^2$, the singular set $\Sigma(F)$ is modified by isotopy, as well as the above listed moves.*

Proof. Let $F\colon M\times[0,1]\to\mathbb{R}^2\times[0,1]$ be a generic homotopy, and let $\pi\colon M\times[0,1]\to[0,1]$ denote the projection to the second factor. If $\pi|_{A_I(F)}$ does not have critical points on the level $M\times\{t_0\}$, then for sufficiently small $\varepsilon>0$, the singular set $A_I(f_t)$, parametrized by $t\in[t_0-\varepsilon, t_0+\varepsilon]$, is modified by an ambient isotopy. Thus, it remains to study modifications of the singular set of f_t corresponding to critical points of the generic functions $\pi|_{A_I(F)}$. We claim that $\pi|_{A_I(F)}$ has no critical points when $I=\{1\}$.

Lemma 4.3. *The map $\pi|_{A_1(F)}$ is a submersion.*

Proof. Over the set $A_1(F)$ of critical points, there is a well-defined kernel bundle $K_1(F)$ of dF. In fact, over $A_1(F)$ there is a splitting

$$T(M\times[0,1])|_{A_1(F)} \cong K_1(F)|_{A_1(F)} \oplus TA_1(F).$$

Assume that there is a critical point $p\in A_1(F)$ of the function $\pi|_{A_1(F)}$. Then $T_p(A_1(F))$ is in the kernel of $d_p\pi$. On the other hand, the projection $d_p\pi$ coincides with the composition

$$T_p(M\times[0,1]) \longrightarrow T_{F(p)}(\mathbb{R}^2\times[0,1]) \longrightarrow T_{\pi(p)}([0,1])$$

of d_pF and the differential of the projection $\mathbb{R}^2\times[0,1]\to[0,1]$ onto the second factor. Since $K_1(F)|_p$ is in the kernel of d_pF, it follows that $K_1(F)|_p$ is in the kernel of $d_p\pi$. To summarize, we have shown that $T_p(M\times[0,1])$ is in the kernel of $d_p\pi$, which contradicts the fact that π is a submersion. $\square$

Let us now consider critical points of the function $\pi|_{A_2(F)}$.

Lemma 4.4. *Let $p\in A_2(F)$ be a critical point of $\pi|_{A_2(F)}$. Then p is a critical point of $\pi|_{\Sigma(F)}$.*

Proof. As above, over $A_2(F)$, there is a well-defined kernel bundle $K_1(F)$. Let L denote the vector subbundle of $T(M \times [0,1])|_{A_2(F)}$ given by $K_1(F) \cap T(\Sigma(F))$. It follows that $\dim L = 1$, and there is a splitting

$$T(\Sigma(F))|_{A_2(F)} \cong L \oplus T(A_2(F)).$$

Since L_p belongs to the kernel $K_1(F)|_p$ of $d_p\pi$, it belongs to the kernel of $d_p\pi|_{\Sigma(F)}$. On the other hand, if p is a critical point of $\pi|_{A_2(F)}$, then $T_p(A_2(F))$ is also in the kernel of $d_p\pi|_{\Sigma(F)}$. Thus, the point p is a critical point of $\pi|_{\Sigma(F)}$. □

By Lemma 4.4, if p is a critical point of $\pi|_{A_2(F)}$, then p is also a critical point of the function $\pi|_{\Sigma(F)}$. If the index of the critical point p is 0, then p corresponds to the appearance (birth) of a wrinkle singularity in $\Sigma(f_t)$. A critical point of index 1 corresponds to the cusp merge move or its inverse, while a critical point of index 2 corresponds to the disappearance (death) of a wrinkle singularity.

The critical points of π restricted to the submanifold of double points of $A_{11}(F)$ correspond to Reidemeister-II fold crossings. All points of $A_{12}(F), A_{111}(F)$, and $A_3(F)$ are critical in the sense that the differential of $\pi|_{A_I(F)}$ in these cases vanishes. It remains to observe that points of $A_{12}(F)$ correspond to cusp-fold crossings, $A_{111}(F)$ correspond to Reidemeister-III fold crossings, and $A_3(F)$ correspond to swallowtail singularities. □

Remark 4.5. The counterpart of Lemma 4.3 for a generic concordance

$$F\colon M \times [0,1] \to \mathbb{R}^2 \times [0,1]$$

of smooth maps is not valid. There are moves of generic concordances that do not occur under a generic homotopy. Specifically, under a generic concordance, an embedded circle of fold points may appear or disappear, and the curves of fold points may be modified by embedded surgery of index 1.

5. Oriented abstract singular set diagrams

The proof of the main results relies on so-called abstract singular set diagrams, which we introduce now.

Let S denote a closed (possibly not path-connected) manifold of dimension 1 together with two disjoint families $P \subset S$ and $Q \subset S$, of finitely many distinguished points. We require that the number of points in Q is even, and that the points in Q are paired. We denote the distinguished points in the family P by $p_1, p_2, \ldots$, and the points in Q by $q_1, q_1', q_2, q_2', \ldots$, where the points q_i and q_i' are paired. We say that a compact subset of S is an *arc* if its interior contains no distinguished points, and its boundary is either empty or consists of the distinguished points.

Definition 5.1. An *oriented abstract singular set diagram* consists of the manifold S, the families P and Q, and an orientation of all arcs on S such that

- if two arcs α and β share a common point $p_i \in P$, then the orientations of α and β agree
- if $q_j \in Q$ is a common point of arcs α and β, while $q_j' \in Q$ is a common point of arcs α' and β', then the orientations on α and β agree if and only if the orientations on α' and β' agree.

In the stated requirements, we allow that some of the arcs α, β, α' and β' may coincide. We note that as a point x traverses a path component of S, the orientation of S at x, that agrees with the orientation of an arc containing x, may change only at a point in Q. Furthermore, at a point in Q the orientation of S may or may not change. For the sake of convenience, we will simply refer to an oriented abstract singular set diagram as a *diagram*.

6. Chessboard functions

In order to properly equip a singular set diagram with a so-called canonical local orientation and coorientation, we first need to introduce the concept of a chessboard function. Let $f\colon M \to N$ be a generic smooth map of a closed manifold M of dimension m to an oriented manifold N of dimension n. We say that a curve γ in N is a *generic curve* with respect to $f(\Sigma)$ if it intersects each Thom-Boardman stratum $f(\Sigma^I)$ of the singular set transversely. In particular, we have $\gamma \cap f(\Sigma) = \gamma \cap f(\Sigma^{d+1,0})$, where $d = m - n$ is the dimension of the map f.

If necessary, we can further perturb the generic curve γ, so that it avoids self-intersection points of the immersed fold surface $f(A_1)$.

Definition 6.1. We say that a locally constant function $c : N\backslash f(\Sigma) \to \mathbb{Z}$ (respectively, $c : N\backslash f(\Sigma) \to \mathbb{Z}_2$) is an *integral chessboard function* (respectively, a *$\mathbb{Z}_2$-valued chessboard function*) if the values $c(\gamma(-1))$ and $c(\gamma(1))$ differ by precisely 1 for each generic curve $\gamma\colon [-1, 1] \to N$ intersecting $f(\Sigma)$ at a unique point $\gamma(0)$.

We say that a singular value y of a map f is a *simple singular value* if the fiber $f^{-1}(y)$ contains a unique singular point. We note that for a generic smooth map, the submanifold of N of simple fold values is dense in $f(\Sigma)$. A *local orientation* of $f(\Sigma)$ is an orientation of the submanifold of simple fold values. Similarly, a *local coorientation* of $f(\Sigma)$ is a coorientation in N of the submanifold of simple fold values. We say that a local orientation of $f(\Sigma)$ agrees with the local coorientation of $f(\Sigma)$ if the local orientation of $f(\Sigma)$ followed by the local coorientation of $f(\Sigma)$ agrees with the standard orientation of N.

Definition 6.2. An integral (respectively, $\mathbb{Z}_2$-valued) chessboard function c defines a *canonical local coorientation* on $f(\Sigma)$ in the direction of the region over which c assumes the smaller value (respectively, the even value). The local orientation that agrees with the canonical local coorientation is said to be a *canonical local orientation*.

Let $f\colon M \to F$ be a stable map of a closed manifold of dimension $m \geqslant 2$ to an oriented surface F, and c a chessboard function, either integral or $\mathbb{Z}_2$-valued. Then, the pair (f, c) gives rise to a diagram $(\Sigma(f); P, Q)$, where $\Sigma(f)$ is the singular set of the map f, and the subsets P and Q of distinguished points are the sets $A_2(f)$ and $A_{11}(f)$ respectively. The pairs (q_i, q_i') of points in Q are the fold points with the same image in N, i.e. the self-intersection points. Finally, the orientation of the arcs of $\Sigma(f)$ is the one induced by the canonical local orientation of $f(\Sigma)$.

Proposition 6.3. *Let $f\colon M \to F$ be a generic map of non-negative dimension of a closed manifold M to an oriented surface F, and c an integral or $\mathbb{Z}_2$-valued chessboard function on $F\backslash f(\Sigma)$. Then $(\Sigma(f); P, Q)$ is an oriented singular set diagram, where $\Sigma(f)$ is equipped with the canonical local orientation.*

Proof. Given a generic map $f\colon M \to F$ of a manifold M, we have defined a manifold $\Sigma(f)$, together with two families of points P and Q that break $f(\Sigma)$ into canonically oriented arcs. By Lemma 6.4 below, the orientations of arcs that share a common point in P agree. By Lemma 6.6 below, if q_j is a common point of arcs α and β, and q_j' is a common point of arcs α' and β', then the orientations on the arcs α and β agree if and only if the same is true for the arcs α' and β'. Thus, indeed, each generic map f of non-negative dimension, together with a chessboard function, defines an oriented singular set diagram. To complete the proof of Proposition 6.3, it remains to provide proofs of Lemma 6.4 and Lemma 6.6.

Lemma 6.4. *Let α and β be two arcs in $\Sigma(f)$ that share a common endpoint $p \in P$. Then, the canonical orientations of arcs α and β agree.*

Notice that in the statement of Lemma 6.4, we do not require that α and β are distinct.

Proof. Consider a neighborhood $W \subset F$ of the image of a cusp point $p \in P$. We may assume that the curve $(f(\alpha) \cup f(\beta)) \cap W$ splits W into two regions. The coorientation of $f(\alpha)$ and $f(\beta)$ are in the direction of the region where the chessboard function assumes the smaller value for integer chessboard functions, and an even value for $\mathbb{Z}_2$-valued chessboard functions. In particular, these coorientations agree. Thus, the orientations of α and β agree. □

Suppose now that $\alpha, \alpha', \beta, \beta'$ are four arcs in $\Sigma(f)$, such that α and β share a common endpoint $q \in Q$, while α' and β' share a common endpoint $q' \in Q$, where q and q' are paired points, i.e. $y = f(q) = f(q')$. Then, the curve $f(\Sigma) \cap U$ breaks a neighborhood U of y in F into four regions. We call these regions L, T, R, B for left, top, right, and bottom, respectively.

Let (a, b, c, d) be the values of the chessboard function c at four points that are respectively in the regions L, T, R, B, e.g., see Fig. 7. We note that the order of entries (a, b, c, d) depends on the choice of, say, the left region L. However, up to cyclic permutations, the tuple (a, b, c, d) is an invariant of the double point, called the *type* of the double point.

Lemma 6.5. *Each type of double points is either of the form $(a, a+1, a, a-1)$ or $(a, a+1, a, a+1)$ for some a, where a is a non-negative integer if the chessboard function is integral, and it is an element of $\mathbb{Z}_2$ if the chessboard function is $\mathbb{Z}_2$-valued.*

The proof of Lemma 6.5 is straightforward; we omit it. We note that if the chessboard function is $\mathbb{Z}_2$-valued, then the two types of double points in Lemma 6.5 are the same.

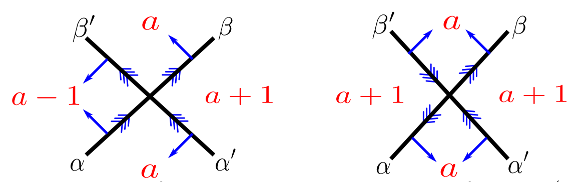

FIGURE 7. Coorientation of arcs near double points of types $(a-1, a, a+1, a)$ on the left, and $(a+1, a, a+1, a)$ on the right.

Lemma 6.6. *The canonical orientations of α and β agree if and only if the canonical orientations of α' and β' agree.*

Proof. We will give an argument for an integral chessboard function; for a $\mathbb{Z}_2$-valued chessboard function the argument is similar. Without loss of generality, we may assume that the arcs are labeled $\alpha, \beta, \alpha', \beta'$ as in Fig. 7. By Lemma 6.5, the type of the double point is either of the form $(a, a+1, a, a-1)$ or $(a, a+1, a, a+1)$. If the double point is of the form $(a, a+1, a, a-1)$, then the values of c are as shown on the left schematic of Fig. 7. Therefore, the coorientations, and hence orientations, of α and β agree. Similarly, the orientations of α' and β' agree. If the double point is of the form $(a, a+1, a, a+1)$, then the values of c are as on the right schematic of Fig. 7, and therefore, the coorientations, and hence orientations, of α and β do not agree. Similarly, the orientations of α' and β' disagree, as well. □

This completes the proof of Proposition 6.3. □

7. Examples of Chessboard functions

In this section we give several examples of integral chessboard functions. We note that the reduction modulo 2 turns any integral chessboard function into a $\mathbb{Z}_2$-valued chessboard function.

7.1. **The chessboard function for maps of dimension 0 counting path components of the fiber.** Let $f\colon M \to N$ be a generic map of a closed manifold of dimension n to an oriented manifold of dimension n. We say that the map f is of *odd degree* if the number of points in the inverse image of any regular value of f is odd. Otherwise, we say that f is of *even degree*. For a regular value $y \in N$ of f, let $\#|f^{-1}(y)|$ denote the number of path components in the fiber $f^{-1}(y)$. Consider the following integer-valued function:

$$c(y) = \begin{cases} \frac{\#|f^{-1}(y)|}{2} & \text{if } f \text{ is of even degree,} \\ \frac{\#|f^{-1}(y)|+1}{2} & \text{if } f \text{ is of odd degree.} \end{cases}$$

It immediately follows that c is an integral chessboard function.

7.2. **The chessboard function for maps of dimension 1 counting path components of the fiber.** Let $f\colon M \to N$ be a generic map of a closed oriented manifold of dimension $n+1$ to an oriented manifold N of dimension n. For a regular value $y \in N$ of f, let $c(y)$ denote the number of path components in the fiber $f^{-1}(y)$, i.e.

$$c(y) = \#|f^{-1}(y)|$$

We claim that $c(y)$ is a chessboard function on $N \backslash f(\Sigma)$. Indeed, let z be a fold singular value of f that is not a self-intersection point of $f(\Sigma)$. Then there is a disc neighborhood $U \ni z$ such that $U \backslash (U \cap f(\Sigma))$ consists of two open discs U_1 and U_2.

Lemma 7.1. *Suppose that $f : M \to N$ is a generic map of an oriented closed manifold of dimension $n+1$ to an oriented manifold N of dimension n. Let $y_1 \in U_1$ and $y_2 \in U_2$ be two points. Then, the number of path components in the fiber $f^{-1}(y_1)$ differs from the number of path components in the fiber $f^{-1}(y_2)$ precisely by 1, i.e.*

$$\#|f^{-1}(y_2)| = \#|f^{-1}(y_1)| \pm 1$$

Proof. Without loss of generality, we may assume that $U \cong D^{n-1} \times (-1,1)$, while $f(\Sigma) \cap U$ coincides with $D^{n-1} \times \{0\}$, where D^{n-1} is a disc of dimension $n-1$. Let γ denote the embedded curve $\{0\} \times (-1,1)$. We may assume that y_1 and y_2 are points on γ. Now, let $\pi_1 : U \to D^{n-1}$ denote the projection of U onto the first factor. Then the composition $\pi_1 \circ f|_{M_0} \colon f^{-1}(U) \to D^{n-1}$ is a proper submersion of the manifold $M_0 := f^{-1}(U)$. Indeed, since both $f|M_0$ and π_1 are proper, we deduce that their composition is also proper. To show that f is a submersion we need to show that the homomorphism $d_x(\pi_1 \circ f|_{M_0})$ is surjective for each point $x \in M_0$. We have

$$\operatorname{Im} d(\pi_1 \circ f|_{M_0}) = d\pi_1(\operatorname{Im} d(f|_{M_0})).$$

If $x \in M_0$, then both $d_x f$ and $d_{f(x)}\pi_1$ are surjective, and therefore their composition $d_x(\pi_1 \circ f|_{M_0})$ is also surjective. If $x \in \Sigma(f)$, then the image of $d_x f$ is $T_{f(x)}(D^{n-1} \times \{0\})$. The restriction of $d\pi_1$ to this space is a surjective map. Thus, again the homomorphism $d_x(\pi_1 \circ f|_{M_0})$ is surjective. Consequently, the map $\pi_1 \circ f|_{M_0}$ is a trivial fiber bundle with fiber diffeomorphic to $V := f^{-1}(\gamma)$, i.e. $M_0 \cong V \times D^{n-1}$. In view of the inherited orientation on M_0, we deduce that the manifold V is also orientable.

Now, we examine the number of components of the preimages $f^{-1}(y_1)$, $f^{-1}(y_2)$ which are subsets of the surface V. Since the restriction $f|_V \colon V \to \gamma$ is a Morse function, the manifold $f^{-1}(y_2)$ is obtained from $f^{-1}(y_1)$ by an elementary oriented surgery. We conclude that the numbers of path components in $f^{-1}(y_1)$ and $f^{-1}(y_2)$ differ by exactly 1. ☐

Lemma 7.1 shows that the function c counting the number of path components in the regular fibers of f is an integral chessboard function. In particular, the image of the singular set $f(\Sigma)$ carries a canonical local coorientation.

7.3. **The Euler chessboard function.** Let $f\colon M \to N$ be a generic map of a closed manifold of dimension $n+2q$ for some $q \geqslant 0$ to an oriented manifold N of dimension n. Let c be the following continuous integer valued function on $\mathbb{R}^n \backslash f(\Sigma)$:

$$c(y) = \begin{cases} \frac{\chi(f^{-1}(y))}{2} & \text{if } \chi(f^{-1}(y)) \text{ is even,} \\ \frac{\chi(f^{-1}(y))+1}{2} & \text{if } \chi(f^{-1}(y)) \text{ is odd.} \end{cases}$$

Recall that under elementary surgery, the Euler characteristic of fibers in adjacent regions is changed by ± 2. From this fact, it follows that c is in integral chessboard function.

7.4. **The depth function.** Let $f\colon M \to \mathbb{R}^n$ be a generic smooth map of a closed manifold, where $n > 1$. Given a point $y \in \mathbb{R}^n \backslash f(\Sigma)$, we say that a path γ is a *path to infinity* if one endpoint of γ is contained in the unbounded region of $\mathbb{R}^n \backslash f(\Sigma)$. We note that since $n > 1$, the unbounded region is unique. Also, we say that a path ℓ_y from y to infinity is a *generic curve* if it intersects each stratum $f(\Sigma^I)$ of the singular set transversely, and the intersection $\ell_y \cap f(A_{11})$ is empty. We note that a generic curve ℓ_y is disjoint from the strata $f(\Sigma^I)$ of dimension $\leqslant n-2$. Consequently, the curve ℓ_y only intersects the singular set $f(\Sigma)$ at fold critical values, i.e., the intersection $\ell_y \cap f(\Sigma)$ is a subset of $f(A_1)$.

The depth function $d\colon \mathbb{R}^n \backslash f(\Sigma) \to \mathbb{Z}_{\geqslant 0}$ associates with each point y the minimal number of intersection points $\ell_y \cap f(\Sigma)$, where ℓ_y ranges over all generic paths from y to infinity. For estimates of the invariant

$$\operatorname{dep}(\Sigma) = \min\{d(y) \mid y \in \mathbb{R}^n \backslash f(\Sigma)\}$$

we refer the reader to [7].

Lemma 7.2. *Let $f\colon M \to \mathbb{R}^n$ be a smooth generic map of a closed manifold of dimension $m \geqslant n$. Suppose that $n > 1$. Let $\gamma\colon [-1,1] \to \mathbb{R}^n$ be a smooth embedded curve with image in $(\mathbb{R}^n \backslash f(M)) \cup f(A_0) \cup f(A_1)$. Suppose that γ intersects $f(A_1)$ transversely at a unique point $\gamma(0)$, and define $y = \gamma(1)$ and $z = \gamma(-1)$. Then $d(y) = d(z) \pm 1$.*

Proof. Let X denote the set of singular points $x \in \Sigma^I(f)$ of types $I = (m-n+1, 0)$ and $(m-n+1,1,0)$. Then $f(X) \subset \mathbb{R}^n$ is a submanifold of codimension 1 with cusps and $f(\Sigma \backslash X)$ is a finite union of submanifolds of $\mathbb{R}^n$ of codimension at least 3. Indeed, the set $\Sigma(f)$ is the union of sets $\Sigma^i(f)$, which consist of points x at which the kernel rank is i, where $i = m-n+1, ..., m$. If f is generic, then each $\Sigma^i(f) \subset M$ is a submanifold of codimension $i(n-m+i)$. In particular, if $i \geqslant m-n+2$, then the codimension of $\Sigma^i(f)$ is at least 4. Similarly, by the Boardman formula [2, §2.5], the codimension of $\Sigma^{i_1, i_2, ..., i_k}(f)$ is

$$\nu_{i_1,...,i_k}(m,n) = (n-m+i_1)\mu(i_1,...,i_k) - (i_1-i_2)\mu(i_2,...,i_k) - ... - (i_{k-1}-i_k)\mu(i_k),$$

where $\mu(i_1,...,i_k)$ is the number of sequences $j_1, .., j_k$ of non-negative integers such that $j_1 \geqslant j_2 \geqslant ... \geqslant j_k$, and $i_1 \geqslant j_1 > 0$, $i_2 \geqslant j_2$, ..., $i_k \geqslant j_k$. Thus, the codimension of a singular stratum $f(\Sigma^I) \subset \mathbb{R}^n$ is at most 2 if and only if I is $(m-n+1,0)$, or $(m-n+1,1,0)$.

Now, let $\gamma \subset \mathbb{R}^n$ be a closed curve intersecting $f(\Sigma)$ transversely at a unique point. Assume, contrary to the conclusion of Lemma 7.2, that $y = d(\gamma(-1))$ does not differ from $z = d(\gamma(1))$ by 1. Let ℓ_y and ℓ_z be respective paths from y and z to infinity that intersect $f(\Sigma)$ transversely precisely $d(y)$ and $d(z)$ times. Without loss of generality, we may assume that the path $\ell_y^{-1} * \gamma * \ell_z$ is closed, where $*$ is path concatenation. It is important to note that this closed path is null-homotopic. Furthermore, without loss of generality, we may assume that $\ell_y^{-1} * \gamma * \ell_z$ avoids $f(\Sigma \backslash X)$ for all moments of time during the homotopy to a point. Thus, under the specified generic homotopy of $\ell_y^{-1} * \gamma * \ell_z$, the number of intersection points of $\ell_y^{-1} * \gamma * \ell_z$ with the stratified manifold $f(X)$ changes by an even number as generically it changes only under finger moves and their inverses. Therefore, the number $d(y)+d(z)+1$ of intersection points of $\ell_y^{-1} * \gamma * \ell_z$ with $f(\Sigma)$ is even. On the other hand, by definition of the depth function, it is clear that $d(y)$ differs from $d(z)$ by at most 1. Thus, $d(y)$ differs from $d(z)$ precisely by 1. □

The depth function can also be defined for any smooth generic map $f\colon M \to N$ of a closed manifold of dimension m to a pointed manifold of dimension $n \leqslant m$. In this case, a path to infinity is a path γ with an endpoint at the distinguished point of N.

We note that the proof of Lemma 7.2 remains valid for maps to simply connected manifolds N.

8. The cumulative winding number

We first recall the definition of the Gauss map. A *parallelized manifold* is a manifold N of dimension n together with a smooth map $\tau\colon TN \to \mathbb{R}^n$ that restricts to an isomorphism $T_xN \to \mathbb{R}^n$ for each points $x \in N$. Given an immersion $\gamma\colon [a,b] \to F$ of a segment into a parallelized surface, the *Gauss map* $G\colon [a,b] \to S^1$ associates with a point $t \in [a,b]$ the unit vector $\tau(\dot\gamma(t))/|\tau(\dot{\gamma(t)})|$. Let $\mathbb{R} \to S^1 = [0,1]/_\sim$, where $\{0\} \sim \{1\}$, be the universal covering that takes a point x to its congruence class modulo 1. Let $\tilde G$ denote a lift of G with respect to the universal covering. We define the *winding number* of γ by $\tilde G(b) - \tilde G(a)$. Given two parametrizations γ' and γ of the same immersed curve, it follows that the winding numbers of γ' and γ are the same if and only if the orientations of the curve induced by γ and γ' agree.

Let f be a generic map to a parallelized surface F. Given a chessboard function c on $F\backslash f(\Sigma)$, let $(\Sigma(f); P, Q)$ denote the diagram associated with f. Let α be an arc of the diagram $(\Sigma(f); P, Q)$. It corresponds to an arc $\bar\alpha = f(\alpha)$ contained in the set $f(\Sigma)$. The curve $\bar\alpha$ is an immersed curve in F, with possible self-intersection points only on the boundary. By definition, the *winding function* φ is a function on the set of arcs of $f(\Sigma)$ that associates with an arc α the winding number $\varphi(\alpha)$ of the curve $\bar\alpha$.

Definition 8.1. Suppose that at every self-intersection point of $f(\Sigma)$ the two intersecting segments are perpendicular. Then the real number

$$\omega(f) := \sum_{\alpha} \varphi(\alpha)$$

is the *cumulative winding number* of $f(\Sigma)$, where α ranges over all arcs of $\Sigma(f)$.

Proposition 8.2. *For a generic smooth map $f : M \to F$ of a closed manifold M of dimension $m \geqslant 2$ to a parallelized surface F, we have*

$$\omega(f) \in \frac{1}{2}\mathbb{Z}.$$

To prove Proposition 8.2 we introduce the notion of a regularization of the singular set. The regularization of the singular set $f(\Sigma)$ is a smooth embedded closed curve $\Re f(\Sigma) \subset F$ obtained from $f(\Sigma)$ by smoothing the curve $f(\Sigma)$ near the cusp points as in Fig. 8, and modifying $f(\Sigma)$ near its self-intersection points. Namely, let y be a self-intersection point of $f(\Sigma)$. Then near y the curve $f(\Sigma)$ consists of four arcs α, α' and β, β'. We remove the four arcs α, β, α' and β' from $f(\Sigma)$ and attach two new arcs so that the orientation on $f(\Sigma)\backslash\{\alpha \cup \beta \cup \alpha' \cup \beta'\}$ extends over the new attached arcs, see Fig. 9, 10, and 11.

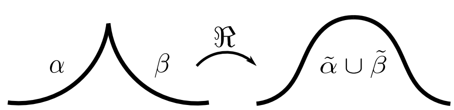

Figure 8. Regularization of a Cusp

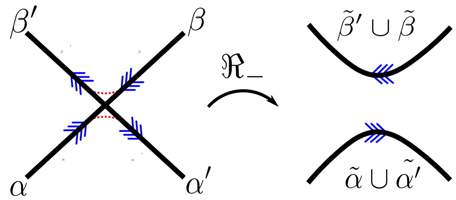

FIGURE 9. Negative Regularization of a Double Point of the form $(a, a+1, a, a+1)$

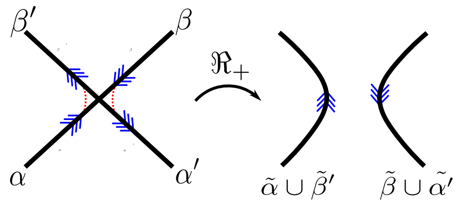

FIGURE 10. Positive Regularization of a Double Point of the form $(a, a+1, a, a+1)$

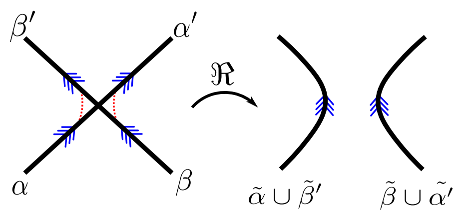

FIGURE 11. Regularization of a Double Point of the form $(a, a+1, a, a-1)$

The proof of the following lemma is omitted as it is straightforward.

Lemma 8.3. *The regularization of a cusp decreases the cumulative winding number by $\frac{1}{2}$ if the coorientations of α and β are as indicated in Fig. 16, and increases the cumulative winding number by $\frac{1}{2}$, otherwise.*

Lemma 8.4. *For a self-intersection point of $f(\Sigma)$ of the form $(a, a+1, a, a+1)$, there are two regularizations that preserve the orientation of the diagram: $\Re_-$ and $\Re_+$. The regularizations $\Re_-$ and $\Re_+$ decrease and increase the cumulative winding number by $\frac{1}{2}$ respectively. For a self-intersection point of the form $(a, a+1, a, a-1)$, the only possible regularization does not change the cumulative winding number.*

Proof. For a self-intersection point of the form $(a, a+1, a, a-1)$, the only possible regularization does not change the cumulative winding number, see Fig. 11. If a double point is of the form $(a, a+1, a, a+1)$ there are two possible regularizations that preserve orientation. One of the regularizations increases the cumulative winding number by $1/2$, while the other one decreases the cumulative winding number by $1/2$, see Fig. 9, and Fig. 10. The two regularizations are denoted by $\Re_+$ and $\Re_-$ respectively. $\square$

Proof of Proposition 8.2. We note that $\Re f(\Sigma)$ consists of embedded curves, and therefore its cumulative winding number is an integer. On the other hand, under the regularization, the cumulative winding number is changed by $\pm\frac{1}{2}$ for each regularization of a cusp, and $\pm\frac{1}{2}$ or 0 for each regularization of a self-crossing. $\square$

9. Changes of the cumulative winding number under homotopy

We now observe and record how the cumulative winding number is changed under generic homotopy. We will denote an R_2 move by $R_2(a_1, a_2, a_3, a_4)$, where the quadruple (a_1, a_2, a_3, a_4) encodes the type of the two self-intersection points that are either being created or removed as a result of the R_2 move. For the remainder of our discussion, we adopt the convention that a_1 corresponds to the bounded region. In Fig. 12, this is the region bounded by $\alpha_2 \cup \beta_2$.

Lemma 9.1. *Let $f\colon M \to F$ be a generic map of a smooth closed manifold of dimension $\geqslant 2$ to a parallelized surface. For any integral chessboard function, there are at most five possible types of R_2 moves: $R_2(a, a-1, a-2, a-1)$, $R_2(a, a+1, a+2, a+1)$, $R_2(a, a+1, a, a-1)$, $R_2(a, a+1, a, a+1)$, and $R_2(a, a-1, a, a-1)$. The moves $R_2(a, a-1, a-2, a-1)$, $R_2(a, a+1, a+2, a+1)$ and $R_2(a, a+1, a, a-1)$ do not change ω. The moves $R_2(a, a+1, a, a+1)$ and $R_2(a, a-1, a, a-1)$ change the cumulative winding number by 1 and -1 respectively.*

Proof. Consider an R_2-move of type $R_2(a_1, a_2, a_3, a_4)$. Since the numbers a_i represent the values of an integral chessboard function, we have $a_{i+1} = a_i \pm 1$ and $a_4 = a_1 \pm 1$. Since up to rotation, the type $R_2(a, a-1, a, a+1)$ is the same as $R_2(a, a+1, a, a-1)$, the list of R_2 moves in the statement of Lemma 9.1 exhausts all possibilities of different types of R_2 moves.

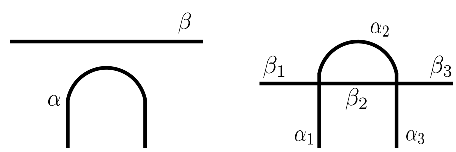

Figure 12. Labeled arcs before and after an R_2 move

It remains to compute the changes of the cumulative winding number ω under each R_2 type move. Denote the two arcs undergoing an R_2 move by α and β. Without loss of generality, we assume that β is straight and fixed, so that only α moves under homotopy. After the R_2 move, the two new double points partition the diagram into six arcs: $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$, and β_3 (see Fig 12). We notice that for any type of R_2 move, the winding numbers of $\beta, \alpha_1, \alpha_3, \beta_1, \beta_2$, and β_3 are trivial. Thus, the change in the cumulative winding number ω is the same as the difference of the winding numbers of α and α_2. For example, for the move $R_2(a, a-1, a-2, a-1)$, the winding numbers $\varphi(\alpha)$ and $\varphi(\alpha_2)$ are $-1/2$. Therefore, the cumulative winding number does not change under the R_2 move of type $R_2(a, a-1, a-2, a-1)$. The changes in the cumulative winding number for the other R_2 moves can be calculated similarly. $\square$

Next we turn to the case of swallowtail moves. Denote the swallowtail move that creates a self-intersection point of type (a_1, a_2, a_3, a_4) by $ST(a_1, a_2, a_3, a_4)$, where a_1 corresponds to the bounded region. In Fig. 13, this is the region entrapped by $\alpha_1 \cup \alpha_2 \cup \alpha_3$.

For a $\mathbb{Z}_2$-valued chessboard function, we say that an R_2 move or an ST move is *even* if the value of the chessboard function over the bounded region is 0. Otherwise, we say that the R_2 move or ST move is *odd*.

Lemma 9.2. *For any $\mathbb{Z}_2$-valued chessboard function, there are at most two R_2 moves: even and odd. An even R_2 move increases the cumulative winding number by 1, while an odd R_2 move decreases the cumulative winding number by 1.*

We omit the proof of Lemma 9.2 since the proofs for even and odd R_2 moves are the same as those for $R_2(a, a+1, a, a+1)$ and $R_2(a, a-1, a, a-1)$ in Lemma 9.1.

Lemma 9.3. *Let $f\colon M \to F$ be a generic map of a smooth closed manifold of dimension $\geqslant 2$ to a parallelized surface. For any integral chessboard function there are at most four* *possible types of swallowtail moves, namely,* $ST(a, a+1, a+2, a+1)$*,* $ST(a, a+1, a, a+1)$*,* $ST(a, a-1, a, a-1)$*, and* $ST(a, a-1, a-2, a-1)$*. Moreover, the moves* $ST(a, a+1, a+2, a+1)$ *and* $ST(a, a-1, a-2, a-1)$ *do not change the cumulative winding number. The moves* $ST(a, a+1, a, a+1)$ *and* $ST(a, a-1, a, a-1)$ *respectively decrease and increase the cumulative winding number by* $\frac{1}{2}$*.*

Proof. Given a swallowtail type $ST(a_1, a_2, a_3, a_4)$, the numbers a_i represent the values of a chessboard function and therefore satisfy the relations $a_{i+1} = a_i \pm 1$ and $a_4 = a_1 \pm 1$. Consequently, $ST(a, a+1, a+2, a+1)$, $ST(a, a+1, a, a+1)$, $ST(a, a-1, a, a-1)$, and $ST(a, a-1, a-2, a-1)$ are the only possible types of swallowtail moves.

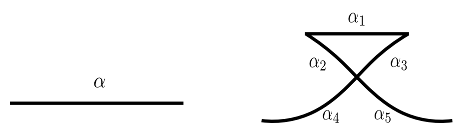

FIGURE 13. Labeled arcs before and after a swallowtail move

We now calculate how the winding number is affected by the swallowtail move of type $ST(a, a+1, a+2, a+1)$. Under such a move, an arc α of the singular set diagram $f(\Sigma)$ is replaced with five sub-arcs: $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and α_5 (see Fig. 13). Without loss of generality, we may assume that α_1 corresponds to the arc whose endpoints are both cusps. We may assume that α and α_1 are straight, thus $\varphi(\alpha) = \varphi(\alpha_1) = 0$. Then $\varphi(\alpha_2) = \varphi(\alpha_3) = -\frac{1}{8}$, $\varphi(\alpha_4) = \varphi(\alpha_5) = \frac{1}{8}$, and therefore the cumulative winding number of the singular set does not change under the swallowtail move of type $ST(a, a+1, a+2, a+1)$.

The change of the cumulative winding number for other types of swallowtail moves can be calculated similarly. □

Lemma 9.4. *For any* $\mathbb{Z}_2$*-valued chessboard function, there are at most two ST moves: even and odd. An even ST move decreases the cumulative winding number by* $1/2$*, while an odd ST move increases the cumulative winding number by* $1/2$*.*

The proofs of Lemma 9.4 are the same as those for $ST(a, a+1, a, a+1)$ and $ST(a, a-1, a, a-1)$ in Lemma 9.3.

It remains to examine how the cumulative winding number $\omega(f)$ is changed under wrinkles, R_3 moves, cusp-fold moves, and cusp merges.

Lemma 9.5. *Let* $f\colon M \to F$ *be a generic map of a smooth closed manifold of dimension* $\geqslant 2$ *to a parallelized surface. For any integral chessboard function, the wrinkle, cusp merge, and cusp-fold moves do not change the cumulative winding number associated with the diagram* $(\Sigma(f); P, Q)$*.*

Proof. The statements of Lemma 9.5 for wrinkles and cusp merges are easily verified. Next, we examine how cusp-fold moves affect ω. Label the arcs before and after a cusp-fold move as in Fig. 14. Then the contribution of $\varphi(\alpha)$ is replaced with $\varphi(\alpha_1)+\varphi(\alpha_2)$, the contribution of $\varphi(\beta)$ is replaced with $\varphi(\beta_1)+\varphi(\beta_2)$, and the contribution of $\varphi(\gamma)$ is replaced with $\varphi(\gamma_1)+\varphi(\gamma_2)+\varphi(\gamma_3)$. Consequently, under a cusp-fold move the winding number is modified continuously. Since the cumulative winding number is an element of $\frac{1}{2}\mathbb{Z}$, we conclude that ω is unchanged under cusp-fold moves. □

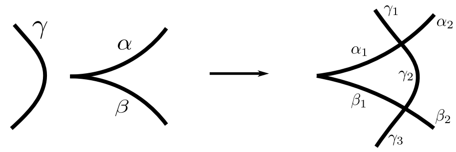

FIGURE 14. Labeling of arcs involved in a cusp-fold move

Similarly, we can determine changes of cumulative winding number for $\mathbb{Z}_2$-valued chessboard functions.

Lemma 9.6. *For $\mathbb{Z}_2$-valued chessboard functions, the cusp merge, cusp-fold, and wrinkle moves do not change the cumulative winding number.*

Lemma 9.7. *For any integral chessboard function, R_3 moves change the cumulative winding number by $\pm 1/2$ or ± 1 or 0. For any $\mathbb{Z}_2$-valued chessboard function, R_3-moves change the cumulative winding number by ± 1.*

The following proposition summarizes the above calculations for $\mathbb{Z}_2$-valued chessboard functions.

Proposition 9.8. *Let $f\colon M \to F$ be a generic map of a smooth closed manifold of dimension $\geqslant 2$ to a parallelized surface. For any $\mathbb{Z}_2$-valued chessboard function, under generic homotopy of a stable map f, the cumulative winding number $\omega(f)$ may change only under an ST, R_2 or R_3 move. Under an ST move, the cumulative winding number changes by $\pm\frac{1}{2}$. Under an R_2 or R_3 move, the cumulative winding number changes by ± 1.*

In the rest of the section we prove Lemmas 9.11 and 9.13. To prove Lemmas 9.11 and 9.13, we will need Lemmas 9.9 and 9.10.

Lemma 9.9. *Lef $f\colon M \to F$ be a smooth map of a closed oriented manifold of dimension 3 to an oriented surface. Then for the integral chessboard function of §7.2, the coorientation of arcs in $(\Sigma(f); P, Q)$ that have a cusp endpoint is as on Fig. 16. The opposite coorientation is not possible.*

Proof. Recall that locally a generic map $f\colon M \to \mathbb{R}^2$ is a Morse 2-function. In particular, for a cusp point $p \in A_2(f)$, we may identify a neighborhood V of $f(p)$ with $[0,1]\times[0,1]$, and the inverse image $f^{-1}(V)$ with $[0,1] \times M_0$ in such a way that $f|_{f^{-1}(V)}$ is given by $(t,x) \mapsto (t, g_t(x))$, where g_t is a family of generalized Morse functions such that g_t has no critical points for $t \in [0, 1/2)$, $g_{1/2}$ has a unique critical point, and g_t has two canceling Morse critical points for $t \in (1/2, 1]$, see Fig. 15.

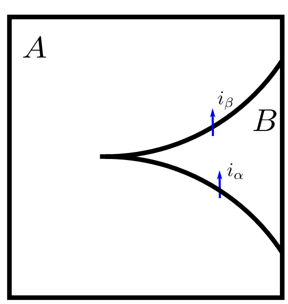

FIGURE 15. The neighborhood V of a cusp point.

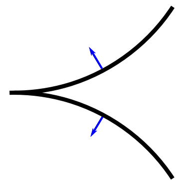

FIGURE 16. Coorientations of singular arcs near a cusp when M is 3-dimensional

Let α and β be two arcs in $f(\Sigma) \cap V$ that share the common cusp endpoint $p \in A_2(f)$. Then the indices i_α and i_β of the two critical points of $g_{3/4}$ on the arcs α and β satisfy the relation $i_\beta = i_\alpha + 1$. The arcs α and β split V into two regions A and B containing the points $(0, 1/2)$ and $(1, 1/2)$ respectively. Both in the case $(i_\alpha, i_\beta) = (0, 1)$ and $(i_\alpha, i_\beta) = (1, 2)$ the number of path components in the inverse image of any point in A is one less than that of any point in B. Therefore, the coorientations of the arcs α and β are as on Fig. 16. □

Lemma 9.10. *Consider a smooth generic map $f : M \to F$ of a closed manifold M of even dimension $m \geqslant 2$ to an oriented surface. When c is the integral Euler chessboard function, $\Sigma(f)$ does not have self-intersection points of type $(a, a-1, a, a-1)$.*

We note that the statement of Lemma 9.10 is not true for the depth chessboard function.

Proof. The intersecting strands of $f(\Sigma)$ break a neighborhood of a self-intersection point into four regions, which we denote by R, T, L and B, for the right, top, left, and bottom regions, respectively. Note that the diffeomorphism types of the fibers M_R, M_T, M_L and M_B over points in the four respective regions do not depend on the choice of regular values. If the manifold M_T is obtained from M_R by a surgery of index i, then M_L is obtained from M_B by a surgery of the same index i. Since M is of even dimension, we conclude

$$\chi(M_T) - \chi(M_R) = \chi(M_L) - \chi(M_B) = \pm 2.$$

This rules out the existence of double points of type $(a, a-1, a, a-1)$. □

Recall that a cusp-fold move creates or eliminates two double points of the same type. We will henceforth denote cusp-fold moves creating or eliminating double points of type (a_1, a_2, a_3, a_4) by $CF(a_1, a_2, a_3, a_4)$, and practice the convention that a_1 corresponds to the value of a prescribed chessboard function in the bounded region (in Figure 14 this is the region with boundary $\alpha_1 \cup \beta_1 \cup \gamma_2$). In particular, there are at most two types of cusp-fold moves involving self-intersection points of type $(a, a-1, a, a-1)$, namely, $CF(a, a-1, a, a-1)$ and $CF(a, a+1, a, a+1)$.

Lemma 9.11. *Let $f, g : M \to F$ be two homotopic image simple maps of a closed manifold M to an oriented surface F.*

- *If M is an oriented manifold of dimension 3, then for the integral or $\mathbb{Z}_2$-valued chessboard function counting path components of fibers, the number of cusp-fold moves involving self-intersection points of type $(a, a-1, a, a-1)$ is even.*
- *If $\pi_1(F) = 1$ and M is of odd dimension $\geqslant 3$, then the number of cusp-fold moves involving self-intersection points of type $(a, a-1, a, a-1)$, with respect to the integral or $\mathbb{Z}_2$-valued depth chessboard function, is even.*
- *If the dimension $m \geqslant 2$ of M is even, then for the integral Euler chessboard function there are no cusp-fold moves involving self-intersection points of type $(a, a-1, a, a-1)$.*

Proof. Suppose M is a closed oriented manifold of dimension 3 equipped with the integral or $\mathbb{Z}_2$-vaued chessboard function counting path components of fibers.

If a cusp fold move $CF(a_1, a_2, a_3, a_4)$ involves a self-intersection point $(a, a-1, a, a-1)$, then (a_1, a_2, a_3, a_4) is obtained from $(a, a-1, a, a-1)$ by a cyclic permutation. In particular, only moves $CF(a,a-1,a,a-1)$ and $CF(a-1,a,a-1,a)$ may involve self-intersection points of type $(a,a-1,a,a-1)$. Furthermore, we claim that the only possible cusp-fold moves involving self-intersection points of type $(a,a-1,a,a-1)$ are $CF(a,a-1,a,a-1)$.

Indeed, if the chessboard function is $\mathbb{Z}_2$-valued, then there are no cusp-fold moves except for $CF(a,a-1,a,a-1)$. Suppose now that the chessboard function is integral. Equip $(\Sigma(f);P,Q)$ with the chessboard function counting the number of path components of regular fibers. By Lemma 9.9, all cusps are cooriented as in Fig. 16, and therefore, the value of the chessboard function over the bounded region is maximal. Thus, indeed, the only possible cusp-fold move involving self-intersection points of type $(a,a-1,a,a-1)$ is $CF(a,a-1,a,a-1)$.

Similarly, the only possible cusp-fold moves involving self-intersection points of type $(a,a-1,a,a-1)$ are $CF(a,a-1,a,a-1)$ in the case of maps $f\colon M\to F$ of a manifold of arbitrary odd dimension $m\geqslant 3$ equipped with the integral depth chessboard function as the value of the depth chessboard function over the bounded region of Fig. 14 is greater than the value over at least over one adjacent region.

On the other hand, every cusp-fold move changes the parity of self-intersection points of the fold curve where one intersecting segment of the fold curve has an odd index while the other one has an even index. No other moves change the parity of the number of such self-intersection points of type $(a,a-1,a,a-1)$. Since $f(\Sigma)$ and $g(\Sigma)$ are embedded, we conclude that the number of $CF(a,a-1,a,a-1)$ moves must be even.

Now, let $f:M\to F$ be a generic map of a manifold M of an arbitrary even dimension $m\geqslant 2$ to an oriented surface. By Lemma 9.10, there are no self-intersection points of type $(a,a-1,a,a-1)$ with respect to the Euler chessboard function, and therefore, there are no cusp-fold moves involving self-intersection points of this type at all. □

Remark 9.12. We note that for an arbitrary chessboard function, its value need not be maximal over the bounded region created by a cusp-fold move. In general, there may possibly be six different types of cusp-fold moves: $CF(a,a-1,a,a-1)$, $CF(a,a-1,a,a+1)$, $CF(a,a-1,a-2,a-1)$, $CF(a,a+1,a,a+1)$, $CF(a,a+1,a,a-1)$, and $CF(a,a+1,a+2,a+1)$.

Lemma 9.13. *Suppose that $f\colon M\to F$ is a generic map of a closed manifold of even dimension to a parallelized surface. Then for the integral Euler chessboard function, the cumulative winding number does not change under homotopy of f.*

Proof. Consider a map f to F. By Lemma 9.10, no double points of type $(a,a-1,a,a-1)$ may occur for the Euler chessboard function. Consequently, the local coorientation of fold arcs defines a global orientation of the curve of fold points as the coorientations of arcs with common endpoints agree, see Fig. 7. Therefore, R_3 moves do not change the cumulative winding number. The cumulative winding number is preserved by R_2 and ST moves by Lemma 9.1 and Lemma 9.3. □

10. Proof of Theorem 1.1 and Theorem 1.2

Theorem 1.1. *Let f and g be two homotopic image simple fold maps from a closed manifold M of even dimension $m\geqslant 2$ to an oriented surface F of finite genus. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.*

Proof. To begin with let us assume that the target surface is $\mathbb{R}^2$. Recall that $\#|\Sigma(f)|$ denotes the number of components of $\Sigma(f)$. Let c be the integral Euler chessboard function as described in §7.3.

By Lemma 9.13,

$$\omega(f)\equiv\omega(g) \pmod 2.$$

Next, utilizing the hypothesis that $f(\Sigma)$ and $g(\Sigma)$ are embedded, we deduce

$$\omega(f)\equiv\#|\Sigma(f)| \pmod 2.$$

Combining the previous congruences yields the desired result

$$\#|\Sigma(f)| \equiv \omega(f) \equiv \omega(g) \equiv \#|\Sigma(g)| \pmod 2.$$

This concludes the proof of Theorem 1.1 in the case of maps to $\mathbb{R}^2$.

Now, suppose that F is a closed surface. Let p be a point in F, away from $f(\Sigma)$. Then the tangent bundle of $F\backslash\{p\}$ is trivial. We fix a trivialization $\tau\colon T(F\backslash\{p\}) \to \mathbb{R}^2$ of the tangent bundle. Then the winding number $\omega(f)$ is well-defined. Under a generic homotopy of f, the curve $f(\Sigma)$ may slide through the point p.

Lemma 10.1. *As the curve $f(\Sigma)$ slides through the point p, the winding number changes by $\pm\chi(F)$, where $\chi(F)$ denotes the Euler characteristic of the surface F.*

Proof. Indeed, let D be a small disc in F centered at p. Recall that for every nowhere zero vector field u over the boundary of D, there is a well-defined winding number which counts the number of rotations of $u(x)$ with respect to a trivialization of the tangent bundle over D as x traverses the boundary of D. It is well-known that the winding number of $u|\partial D$ equals the sum of indices of critical points of any extension of the vector field u over the disc. Now, let v denote a nowhere vanishing vector field $\tau^{-1}(e_1)$ over $F\backslash \operatorname{Int}(D)$ trivializing the tangent bundle of $F\backslash \operatorname{Int}(D)$. It can be extended to a vector field over F, which we still denote by v. The sum of indices of critical points of v is the Euler characteristic of F. Therefore the winding number of $v|\partial D$ with respect to the trivialization of the tangent bundle of D is $\chi(F)$. Consequently, if w is a unit vector field in $TF|\partial D$ that extends to a unit vector field over D, then the winding number of w with respect to the trivialization τ of the tangent bundle of $F\backslash \operatorname{Int}(D)$ is $\pm\chi(F)$.

Suppose $\{f_t\}$ is a generic homotopy of $f = f_0$, parameterized by $t \in [0,1]$ under which $f(\Sigma)$ slides through the point p. Without loss of generality we may assume that $f_0(\Sigma)$ shares a common point with ∂D and has no other common points with D. Then the curve $f_1(\Sigma)$ is regularly homotopic in $F\backslash\{p\}$ to a smoothening of the concatenation of the curves $f_0(\Sigma)$ and ∂D. Thus, up to sign, the difference between the winding numbers of $f_0(\Sigma)$ and $f_1(\Sigma)$ is the Euler characteristic of F. □

Since the surface F is closed and oriented of genus g, we have $\chi(F) = 2-2g$. Thus, the parity of the winding number is well-defined. Consequently, as in the case when the target surface is $\mathbb{R}^2$, we have

$$\omega(f) \equiv \omega(g) \pmod 2.$$

This also shows that for every embedded closed curve γ on an oriented closed surface F, there is a well-defined winding number $\rho(\gamma) \in \mathbb{Z}_2$. The winding number $\rho(\gamma)$ does not depend on the orientation of γ.

Lemma 10.2. *Let γ_1 and γ_2 be two embedded closed curves on an oriented closed surface F. Suppose that γ_1 and γ_2 represent the same homology class in $H_1(F;\mathbb{Z}_2)$. Then*

$$\rho(\gamma_1) - \#|\gamma_1| \equiv \rho(\gamma_2) - \#|\gamma_2| \pmod 2,$$

where $\#|\gamma_i|$ denotes the number of components of γ_i, for $i = 1, 2$.

Proof. We may assume that the surface F is connected.

Recall that an oriented surgery of an embedded closed curve γ is embedded if the base of surgery is an embedded strip whose interior is disjoint from the curve γ, see Fig. 17 and 18. We note that under each oriented embedded surgery the value $\rho(\gamma)$, as well as the modulo two residue class of $\#|\gamma|$, is changed. Thus, the value $\rho(\gamma) - \#|\gamma|$ remains the same.

By performing an appropriate number of elementary surgeries, we may assume γ_1 and γ_2 are path connected closed embedded curves. Since γ_1 and γ_2 represent the same homology class in $H_1(F;\mathbb{Z}_2)$, they are either both separating or non-separating.

If the curves are non-separating, then there is a diffeomorphism φ of the target surface F to itself that takes γ_1 to γ_2. Thus, the parity of $\rho(\gamma_1) - \#|\gamma_1|$ is the same as the parity of $\rho(\gamma_2) - \#|\gamma_2|$.

Next, suppose that the curves γ_1 and γ_2 are separating. Without loss of generality, we may assume that γ_1 and γ_2 are disjoint, since there is a diffeomorphism φ of F such that γ_1 and $\varphi(\gamma_2)$ are disjoint. We may always construct a Morse function h on F such that γ_1 and γ_2 are two regular level sets of F, say $h^{-1}(0) = \gamma_1$ and $h^{-1}(1) = \varphi(\gamma_2)$. Each critical point of h in $h^{-1}[0, 1]$ corresponds to an elementary oriented embedded surgery. The composition of these elementary oriented embedded surgeries takes γ_1 to a curve isotopic to γ_2. As mentioned above, the value of $\rho(\gamma_1)$ and the modulo two residue class $\#|\gamma_1|$ are changed under each elementary oriented embedded surgery. Therefore, the value $\rho(\gamma_1) - \#|\gamma_1|$ is preserved.

In both cases, we have deduced the desired result. $\square$

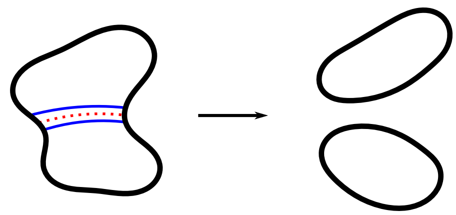

FIGURE 17. Elementary surgery increasing the number of connected components.

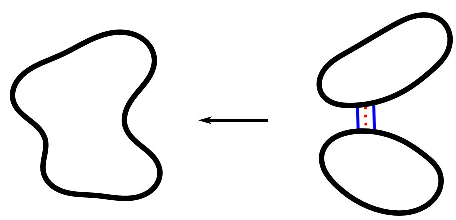

FIGURE 18. Elementary surgery decreasing the number of connected components.

In view of Lemma 10.2, we conclude that

$$\#|\Sigma(f)| \equiv \#|\Sigma(g)| \pmod 2.$$

If F is not a closed surface, then it admits an embedding j into a closed surface F'. Then the numbers of path components of $\Sigma(f)$ and $\Sigma(g)$ are the same as the numbers of components of $\Sigma(j \circ f)$ and $\Sigma(j \circ g)$, respectively. Therefore, the case where F is an open surface of finite genus follows from the case where F is a closed surface. $\square$

Theorem 1.2. *Let f and g be two homotopic image simple fold maps $M \to F$, where*

- *M is a closed manifold of odd dimension $m > 2$ and F is $\mathbb{R}^2$ or S^2, or*
- *M is a closed oriented manifold of dimension 3, and F is an oriented surface.*

Suppose that no R_3 moves occur during a generic homotopy from f to g. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.

Proof. We will work with the $\mathbb{Z}_2$-valued depth chessboard function if $m > 2$ and F is $\mathbb{R}^2$ or S^2. If $m = 3$ and M is oriented, then we will work with the $\mathbb{Z}_2$-valued chessboard function that counts the number modulo 2 of path components in the preimage of a regular value.

Since f and g are odd dimensional image simple fold maps to a surface, the homology class of swallowtail singularities of a homotopy H of f to g is trivial, as the image of the set of swallowtail singular points of H in $F \times [0,1]$ bounds the set of double points of $H(\Sigma)$. Therefore, by the argument in [17], there is a formal homotopy of f to g with no swallowtail singularities. By the relative h-principle for swallowtail singular points [1], we may assume that the (genuine) homotopy of f to g does not have swallowtail singular points.

By Lemma 9.11, the number of cusp-fold moves is even since for $\mathbb{Z}_2$-valued chessboard functions all cusp-fold moves are of type $CF(a, a-1, a, a-1)$.

On the other hand, since there are no swallowtail singular points, the number of pairs of self-intersection points changes under homotopy by

$$\#|CF| + \#|R_2| \equiv 0 \pmod 2.$$

Consequently, the number of R_2 moves is also even.

Suppose now that F is a parallelized surface. By Proposition 9.8, only swallowtail, R_2 and R_3 moves may change the cumulative winding number. We have assumed that the homotopy of f to g does not involve swallowtail and R_3 moves. Therefore, since each R_2 move changes the cumulative winding number by ± 1, and the number of R_2 moves is even, we conclude that the parity of the cumulative winding numbers for f and g are the same. Consequently, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.

The argument in the proof of Theorem 1.1 shows that the same conclusion is true in the case where the target surface F is a sphere if $m > 3$, and in the case where F is an oriented surface of finite genus when $m = 3$. Indeed, in both cases we may choose a trivialization of the tangent bundle of $F \backslash \{p\}$. Therefore, by the argument in the previous paragraph, if $f(\Sigma)$ does not slide through p under the homotopy from f to g, the parities of the cumulative winding numbers for f and g are the same. On the other hand, when $f(\Sigma)$ slides through p, the cumulative winding number changes by an even number $\pm\chi(F)$, by Lemma 10.1. Therefore, the parities of the cumulative winding numbers for f and g are the same. Thus, the parities of $\#|\Sigma(f)|$ and $\#|\Sigma(g)|$ are the same. □

Remark 10.3. We do not know if the statement of Theorem 1.2 is true for arbitrary closed oriented surfaces F when $m > 3$.

11. The invariant I and proof of Theorem 1.3

In this section we prove Theorem 1.3. The main ingredient of the proof is the $\mathbb{Z}_4$-valued homotopy invariant $I(f)$ defined in the introduction. We will recall the precise definition of the function $I(f)$ in the statement of Lemma 11.1.

Let M be a closed manifold of dimension $m \geqslant 2$, and $f\colon M \to F$ a smooth stable map to a surface F. Then, the singular set $\Sigma(f)$ is a closed 1-dimensional submanifold of M, which consists of fold points $A_1(f)$, and finitely many cusp points $A_2(f)$. Recall, the number of components of the singular set $\Sigma(f)$ is denoted by $\#|\Sigma(f)|$, while the number of cusp points is denoted by $\#|A_2(f)|$. We will also consider the number of self-intersection points $\Delta(f)$ of $f(\Sigma)$. We note that if f is generic, then the image of cusp points is not at the self-intersection points of $f(\Sigma)$.

Lemma 11.1. *Let $f, g\colon M \to F$ be two generic maps of a closed manifold of dimension $m \geqslant 2$ into a surface. Suppose that there exists a generic homotopy $H\colon M \times [0,1] \to F \times [0,1]$ between f and g such that the singular set $\Sigma(H)$ is an orientable submanifold of $M \times [0,1]$. Then $I(f) = I(g)$ where*

$$I \equiv \#|A_2| + 2\Delta + 2\#|\Sigma| \pmod 4.$$

Proof. Let $H\colon M \times [0,1] \to N \times [0,1]$ be a generic homotopy such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Under the homotopy H, the singular set of f may be modified by any of the six allowable homotopy moves. Let s denote the number of swallowtail moves and their inverses, and m the number of wrinkles, cusp merges, and the inverses of these moves. R_3 moves do not change the number of self-intersection points. Under R_2 and cusp-fold moves the number of self-intersection points may change, but the congruence class of $2\Delta(f)$ does not change modulo 4. Therefore,

$$2\Delta(g) \equiv 2\Delta(f) + 2s \pmod 4,$$

since every swallowtail move and their inverse changes the number of self-intersection points of the image of the singular set by 1. On the other hand, we have

$$\#|A_2(g)| \equiv \#|A_2(f)| + 2s + 2m \pmod 4,$$

since every swallowtail move, wrinkle, cusp merge and their inverse changes the number of cusps by two. Next, since the singular set of the homotopy H is orientable, every wrinkle, cusp merge and their inverse changes the parity of $\#|\Sigma(f)|$. Consequently, we also have the congruence

$$2\#|\Sigma(g)| \equiv 2\#|\Sigma(f)| + 2m \pmod 4.$$

To summarize,

$$2\#|\Sigma(g)| + 2\Delta(g) + \#|A_2(g)| \equiv 2\#|\Sigma(f)| + 2\Delta(f) + \#|A_2(f)| + 4s + 4m \pmod 4,$$

which simplifies to

$$2\#|\Sigma(g)| + 2\Delta(g) + \#|A_2(g)| \equiv 2\#|\Sigma(f)| + 2\Delta(f) + \#|A_2(f)| \pmod 4,$$

yielding

$$I(g) \equiv I(f) \pmod 4.$$

□

Remark 11.2. When the manifold M is even dimensional and the surface F is orientable, the singular set $\Sigma(f)$ is necessarily orientable, by Theorem 3.4. Thus, the function $I(f)$ is a homotopy invariant for generic maps $f : M \to F$ of a closed manifold of even dimension into an orientable surface.

Corollary 11.3. *The function*

$$I(f) = \#|A_2(f)| + 2\#|\Sigma(f)| \pmod 4$$

is a homotopy invariant of image simple maps $f : M \to F$, where M is an even dimensional closed manifold and F is an orientable surface.

Corollary 11.4. *The function*

$$I(f)/2 = \Delta(f) + \#|\Sigma(f)| \pmod 2$$

is a homotopy invariant of simple stable maps $f : M \to F$, where M is an even dimensional closed manifold and F is an orientable surface.

Theorem 1.3 essentially follows from the existence of the invariant $I(f)$.

Theorem 1.3. *Let f and g be two homotopic image simple fold maps from a closed manifold M of dimension $m \geqslant 2$ to a surface F of finite genus. Suppose the surface $\Sigma(H)$ of singular points of the homotopy H between f and g is orientable. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.*

Proof. Consider the homotopy invariant

$$I(f) = \#|A_2(f)| + 2\Delta(f) + 2\#|\Sigma(f)| \pmod 4.$$

By assumption, the maps f and g have no cusps and are embedded, therefore

$$\#|A_2(f)| = \Delta(f) = 0 \qquad \text{and} \qquad \#|A_2(g)| = \Delta(g) = 0.$$

Therefore,

$$I(f) = 2\#|\Sigma(f)| \pmod 4 \text{ and } I(g) = 2\#|\Sigma(g)| \pmod 4.$$

By Lemma 11.1, we have $I(f) \equiv I(g)$. Thus, $2\#|\Sigma(f)| \equiv 2\#|\Sigma(g)| \pmod 4$, which results in $\#|\Sigma(f)| \equiv \#|\Sigma(g)| \pmod 2$. □

12. Low dimensional applications

In this section we consider examples and applications in the cases of maps to surfaces of manifolds of dimension $m = 2, 3$ and 4.

12.1. **Maps of Surfaces to Surfaces.** Let $f : F_g \to F_h$ be an image simple stable map of oriented surfaces of genera g and h, respectively. By Theorem 1.1, the number of path components in $\Sigma(f)$ depends only on the homotopy class of f. In fact, a stronger statement is true.

Proposition 12.1 (M.Yamamoto [23])**.**

$$\#|\Sigma(f)| \equiv \deg(f)(h-1) - (g-1) \pmod 2.$$

The above proposition holds for arbitrary $g, h \geqslant 0$ and even for non-embedded singular value sets of fold maps. For example, for every fold map f of a sphere into itself (possibly with self-intersecting fold curve $f(\Sigma)$), we have

$$\#|\Sigma(f)| \equiv \deg(f) - 1 \pmod 2.$$

12.2. **Maps of the 3-sphere to the 2-sphere.** In [19], Saeki studied fold maps of 3-dimensional manifolds into surfaces and showed that every closed connected oriented 3-manifold admits a stable map to the 2-sphere without definite fold points. In particular, for maps of the 3-sphere into the 2-sphere, Saeki constructed an image simple indefinite fold map $f : S^3 \to S^2$ such that $\Sigma(f) = n + 1$, where $n \in \mathbb{Z}$ is the Hopf invariant $H(f)$ of f. Saeki posed the following question.

Problem 12.2. *For an integer $n \in \mathbb{Z} \simeq \pi_3 S^2$, let us consider stable maps $f\colon S^3 \to S^2$ without definite fold which represent the associated homotopy class and which satisfies that $\Sigma(f) \neq \varnothing$ and $f|_{\Sigma(f)}$ is an embedding, where $\Sigma(f)$ is the set of singular points of f. Then, is the number of components of $\Sigma(f)$ congruent modulo two to $n+1$?*

Saeki solved Problem 12.2 in the negative in [20]. The following corollary shows under what conditions Saeki's problem can be answered in the positive. As a corollary of Theorems 1.2 and 1.3, we prove the following statement related to Saeki's question.

Corollary 12.3. *Let $f : S^3 \to S^2$ be an image simple indefinite fold map with Hopf invariant $H(f) = n$ constructed by Saeki in [19]. If $g : S^3 \to S^2$ is obtained from f by a homotopy*

$$F : S^3 \times [0,1] \to S^2 \times [0,1]$$

such that $\Sigma(F)$ is orientable or $F(\Sigma)$ has no triple self-intersection points, then

$$\#|\Sigma(g)| \equiv \#|\Sigma(f)| \equiv n + 1 \pmod 2.$$

12.3. **Maps of the 4-sphere to the 2-sphere.** As a consequence of Theorem 1.1, we obtain a result on the 4-dimensional analog of Problem 12.2.

Corollary 12.4. *Let $f : S^4 \to S^2$ be an image simple fold map of the 4-sphere into the 2-sphere. Then*

$$\#|\Sigma(f)| \equiv 1 \pmod 2.$$

Proof. Let us examine an image simple fold map representative of both the trivial and non-trivial elements of $\pi_4(S^2) \cong \mathbb{Z}_2$. We respectively denote the equivalence classes of the trivial and non-trivial elements of $\pi_4(S^2)$ by [0] and [1]. The trivial element is constructed via the standard projection to $\mathbb{R}^2$, followed by the inclusion into S^2, i.e. $f_{[0]} : S^4 \to \mathbb{R}^2 \hookrightarrow S^2$, where $f_{[0]}(\Sigma)$ consists of one closed embedded definite fold curve. Therefore, by Theorem 1.1, any image simple fold map $g \in [0]$ has a singular set such that $\#|\Sigma(g)|$ is odd.

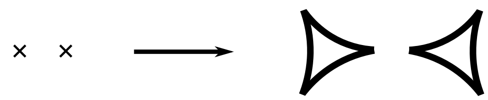

FIGURE 19. Replacing Lefschetz critical points with cusp and indefinite fold points.

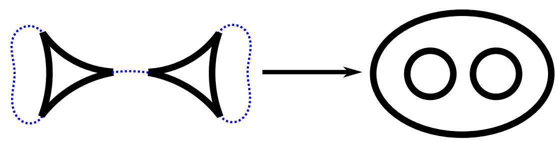

FIGURE 20. Three cusp merges.

Next, we examine the non-trivial element of $\pi_4(S^2)$. Consider the suspension of the Hopf fibration $H : S^3 \to S^2$, defined as $\Sigma H : \Sigma S^3 \to \Sigma S^2$, which is equivalent to $\Sigma H : S^4 \to S^3$. Composition of the suspended Hopf fibration with the Hopf fibration itself results in the map $f_{[1]} : H \circ \Sigma H : S^4 \to S^2$. The singular set of $f_{[1]}$ consists of a pair of Lefschetz critical points, see [13] for a detailed explanation. Each Lefschetz critical point can be deformed into a component consisting of three cusps and indefinite folds as in Figure 19. For an explicit description of the move in Figure 19, we refer the reader to the third section of [11].

We then obtain an embedding of three indefinite fold components after thrice merging pairs of the recently created cusps, see Figure 20. Now, the singular set of the image simple fold map $f_{[1]}$ has an odd number of components and thus, by Theorem 1.1, the singular set of any image simple fold map $h \in [1]$ must also have an odd number of connected components.

Up to homotopy, we have examined the singular set of all image simple fold maps from the 4-sphere to the 2-sphere. In all cases, the singular set has an odd number of connected components. $\square$

Remark 12.5. We note that through steps described in [19], every image simple fold map is homotopic to an image simple indefinite fold map.

Combining the statement of Remark 11.2 with Corollary 12.4, we obtain the following corollary.

Corollary 12.6. *For every smooth stable map $f : S^4 \to S^2$, we have*

$$I(f) \equiv 2 \pmod 4.$$

We may also combine Corollary 11.4 and Corollary 12.6 to get the following result.

Corollary 12.7. *For every simple stable map $f : S^4 \to S^2$, we have*

$$\Delta(f) \equiv \#|\Sigma(f)| + 1 \pmod 2.$$

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