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JET SCHEMES OF SINGULAR SURFACES OF TYPES D_4^0 AND D_4^1 IN CHARACTERISTIC 2

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Abstract. Let k be an algebraically closed field, S a variety over k and m a nonnegative integer. There is a space S_m over S, called the jet scheme of S of order m, parametrizing m-th jets on S. The fiber over the singular locus of S is called the singular fiber.

In this paper, we consider the singular fibers of the jet schemes of 2-dimensional rational double points over a field k of characteristic 2 whose resolution graph is of type D_4. There are two types of such singularities, of type D_4^0 and type D_4^1. We give the irreducible decomposition of the singular fiber and describe the configuration of the irreducible components. The case of a D_4^0-singularity is quite similar to the case of characteristic 0 studied in the previous paper. The case of D_4^1-singularity requires more elaborate analysis of certain subsets of the singular fibers.

1. Introduction

Let k be an algebraically closed field of an arbitrary characteristic and S a surface over k. The notion of a jet scheme was introduced by Nash in 1968 in a preprint, later published in 1995 [9]. Let m be a nonnegative integer. Roughly speaking, an m-th jet of S is an infinitesimal map of order m from a germ of a curve to S, and the scheme parametrizing m-th jets is called the m-th jet scheme. We denote the m-th jet scheme of S by S_m.

For nonnegative integers $m \geq m'$, there is a natural morphism $\pi^S_{m,m'} : S_m \to S_{m'}$ called the truncation morphism. The 0-th jet scheme is identified with S and we denote the truncation morphism $\pi^S_{m,0}$ by π^S_m. We are interested in the fiber S^0_m of π^S_m over the singular locus Sing S of S, which we call the singular fiber of S. In characteristic 0, the relation between the singular fibers of jet schemes of surfaces and the exceptional curves of the minimal resolutions of singularities of surfaces was studied in a series of papers by Mourtada [6, 7] and Mourtada-Plénat [8]. For a general surface S, the relation between the irreducible components of S^0_m and the exceptional curves of the minimal resolution of S is not simple. However, for rational double points, Mourtada gave a one-to-one correspondence between the irreducible components of the singular fiber and the exceptional curves of the minimal resolution.

In [5], this correspondence was studied in more detail. For a fixed order m of the jet scheme of a singular surface of type A_n or type D_4, the configuration of the irreducible components of the singular fiber was investigated. Furthermore, a graph was constructed using this information, and the graph turned out to be isomorphic to the resolution graph of the singularity for a sufficiently large m. Concretely, the graph is constructed as follows: Let $Z^1_m, ..., Z^n_m$ be the irreducible components of the singular fiber S^0_m.

Construction ([5, Construction 2.15])**.** Let $V = \{Z^1_m, ..., Z^n_m\}$, and let $E \subseteq \{Z^i_m \cap Z^j_m \mid i, j \in \{1, ..., n\}, i \neq j\}$ be the set of the maximal elements for the inclusion relation. Then we construct a graph Γ as the pair (V, E), that is, the vertices of Γ are elements of V and there is given an edge between Z^i_m and Z^j_m if and only if $Z^i_m \cap Z^j_m \in E$.

In this paper, we consider 2-dimensional rational double points in positive characteristics, especially, D_4-type singular surfaces in characteristic 2.

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First, we consider the irreducible decomposition of the singular fiber in any characteristic. For singular surfaces of type A_n, the defining equation of the surface is given by $xy - z^{n+1}$ in any characteristics. Here and henceforth, the singular point is at the origin. In this case the irreducible decomposition of the singular fiber was given by Mourtada in [6]. For a singular surface of type D_n for $n \geq 4$, the defining equation is $x^2 - y^2z + z^n$ in characteristic 0. In the first possible case $n = 4$, if the characteristic is a prime different from 2, the defining equation is the same as in characteristic 0. One can check that the arguments in [7] work also in this case.

In characteristic 2, however, there are two different types of singularities with the resolution graph of type D_4. In Artin [1], one is given by $x^2 + y^2z + yz^2$, called a singularity of type D_4^0, and the other given by $x^2 + y^2z + yz^2 + xyz$, called a singularity of type D_4^1. We note that the equation of a singularity of type D_4^0 is weighted homogenous while that of a singularity of type D_4^1 is not. In Mourtada [7], it was important in giving the irreducible decomposition of the singular fibers that the defining equation is weighted homogenous. Thus, for a singular surface of type D_4^0, we will give the irreducible decomposition of the singular fiber using arguments as in Mourtada [7]. On the other hand, for a singular surface of type D_4^1, we have to find another way to prove that certain sets are irreducible. In this paper, we do this by a careful study of the codimensions of certain subsets of the singular fiber.

Second, as for the configuration of the irreducible components of the singular fiber, we consider the inclusion relations among their intersections as in [5]. For the singular surfaces of type A_n, we gave the irreducible decomposition of the intersections of the irreducible components of the singular fiber in characteristic 0 in [5]. These decompositions are independent of the characteristics, so the inclusion relations are the same as in [5] in any characteristic. Next, we consider the singular surfaces of type D_4. In this case, while Mourtada gave the irreducible decomposition of the singular fiber in characteristic 0 in [7], generators of defining ideals of irreducible components were not known. Notwithstanding, in characteristic 0, we could determine the set E, and hence the graph Γ, in Construction in [5]. If the characteristic is greater than 2, we can determine the configuration using the same arguments as in [5] and we do not deal with these cases in this paper. If the characteristic is 2, the arguments need some modification, and this is the case that we will deal with in this paper.

The following two theorems are the main results in this paper. First, we give the irreducible decomposition of the singular fiber.

Theorem 1.1. *Let k be an algebraically closed field of characteristic 2 and $S \subset \mathbb{A}^3$ the surface defined by $f = x^2 + y^2z + yz^2$ or $g = x^2 + y^2z + yz^2 + xyz$ in the affine space over k. If $m \geq 5$, the irreducible decomposition of the singular fiber S_m^0 is given by*

$$S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3,$$

where Z_m^0, Z_m^1, Z_m^2 and Z_m^3 are as in Definition 3.6 or Definition 4.7.

Second, we describe the inclusion relations between the intersections of irreducible components of the singular fiber.

Theorem 1.2. *Let k be an algebraically closed field of characteristic 2, $S \subset \mathbb{A}^3$ the surface defined by $f = x^2 + y^2z + yz^2$ or $g = x^2 + y^2z + yz^2 + xyz$ in the affine space over k, $Z_m^0, \ldots, Z_m^3$ the irreducible components of the singular fiber S_m^0 as in Definition 3.6 or Definition 4.7.*

(a) *For $0 \leq i < j \leq 3$, $Z_m^i \cap Z_m^j \subsetneq Z_m^0$.*
(b) *For $1 \leq i, j \leq 3$ with $i \neq j$, $Z_m^0 \cap Z_m^i \not\subseteq Z_m^0 \cap Z_m^j$.*
(c) *For $1 \leq i, j \leq 3$ with $i \neq j$, $Z_m^i \cap Z_m^j \subsetneq Z_m^0 \cap Z_m^i$.*
(d) *For $1 \leq i < j \leq 3$ and $1 \leq l \leq 3$, $Z_m^0 \cap Z_m^l \not\subseteq Z_m^i \cap Z_m^j$.*

In particular, for $0 \leq i < j \leq 3$, $Z_m^i \cap Z_m^j$ is maximal in $\{Z_m^i \cap Z_m^j \mid i, j \in \{0, 1, 2, 3\}, i \neq j\}$ with respect to the inclusion relation if and only if $(i, j) = (0, 1), (0, 2), (0, 3)$.

The next step would be the case of type D_n with $n \geq 5$, but the calculation becomes more and more difficult. For instance, for $n = 5$, the author could calculate the irreducible components of S_m^0 only for

$m \leq 8$ in characteristic 0 using Macaulay2 on a personal computer. In characteristic 2, the calculations become somewhat simpler, and we hope that the case of characteristic 2 will give some insight into the characteristic 0 case.

The organization of this paper is as follows. In section 2, we fix some notations on jet schemes and rational double points in characteristic 2. In section 3 and section 4, we give a description of the defining ideals of the irreducible components of the singular fibers of the jet schemes of type D_4^0- and D_4^1-singularities, and we study the intersections of the irreducible components to determine the inclusion relations between them. Then we see that the graphs constructed as in Construction 3.17 are isomorphic to the resolution graph of the singularity for a sufficiently large m.

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2. Preliminaries

In this section, we recall the definition of the jet schemes and the defining equations of rational double points in $\mathbb{A}^3$ whose resolution graphs are of type D_4, and fix some notations.

First of all, we recall the definition of the jet schemes. For more details on jet schemes, we refer to [2]. Let k be an algebraically closed field of an arbitrary characteristic, X a scheme of finite type over k and m a nonnegative integer. We consider the following functor F_m^X. Let $\mathfrak{Sch}/k$ be the category of schemes over k and $\mathfrak{Set}$ the category of sets. The functor F_m^X is given by

$$F_m^X : \mathfrak{Sch}/k \to \mathfrak{Set}; Z \mapsto \mathrm{Hom}_k(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/\langle t^{m+1}\rangle, X).$$

The functor F_m^X is representable, and the object representing F_m^X will be denoted by X_m ([2, Theorem 2.2], [3, Proposition 2.2]).

Definition 2.1. The scheme X_m is called the m-th *jet scheme* of X.

We are interested in a neighborhood of two dimensional isolated hypersurface singularity, so we consider an affine scheme embedded in $\mathbb{A}^3$ as the target space. In this case, we have the following explicit description. Let us consider a scheme X which is embedded as a hypersurface in an affine space $\mathbb{A}^3$. Then the affine coordinate ring $\Gamma(X, \mathcal{O}_X)$ of X can be written in the form $k[x, y, z]/\langle f\rangle$. We introduce some notations.

Notation 2.2. Let $R_m := k[x_0, ..., x_m, y_0, ..., y_m, z_0, ..., z_m]$, $\mathbf{x} := x_0 + x_1 t + \cdots + x_m t^m$, $\mathbf{y} := y_0 + y_1 t + \cdots + y_m t^m$, $\mathbf{z} := z_0 + z_1 t + \cdots + z_m t^m \in R_m[t]/\langle t^{m+1}\rangle$. For a polynomial $f \in k[x, y, z]$, we expand $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = f^{(0)} + f^{(1)} t + \cdots + f^{(m)} t^m$$

in $R_m[t]/\langle t^{m+1}\rangle$, where $f^{(j)} \in R_m$. Then the m-th jet scheme X_m can be written as

$$X_m = \mathrm{Spec}\, R_m/\langle f^{(0)}, ..., f^{(m)}\rangle.$$

For a closed point $\gamma = (a_0, ..., a_m, b_0, ..., b_m, c_0, ..., c_m) \in \mathbb{A}^{3(m+1)} = \mathrm{Spec}\, R_m$, we also write

$$\gamma = (\sum_{i=0}^{m} a_i t^i, \sum_{i=0}^{m} b_i t^i, \sum_{i=0}^{m} c_i t^i).$$

Remark 2.3. In this notation, $X_0 = \mathrm{Spec}\, R_0/\langle f^{(0)}\rangle = \mathrm{Spec}\, k[x_0, y_0, z_0]/\langle f^{(0)}\rangle$. Thus, $X_0 \cong X$ and so we identify X_0 with X in the following.

We note the following fact.

Remark 2.4. If we give the weight i to the variables x_i, y_i, z_i for $0 \leq i \leq m$, then the polynomial $f^{(n)}$ is homogenous of degree n for $0 \leq n \leq m$. Indeed, each term of $\mathbf{x}, \mathbf{y}$ or $\mathbf{z}$ has the same degree in x_i, y_i, z_i and in t, and so the same holds for $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Moreover, $f^{(n)}$ is the coefficient of t^n in $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$, hence the claim.

In the following sections, we consider reductions of the polynomials $f^{(i)}$ modulo the ideals generated by x_j's, y_j's and z_j's for different ranges of j. Hence we set up the following notation.

Notation 2.5. Let $m, p, q, r \in \mathbb{Z}_{>0}$. For $p, q, r \leq m+1$, we set
$$L_{pqr} = \langle x_0, ..., x_{p-1}, y_0, ..., y_{q-1}, z_0, ..., z_{r-1} \rangle \subset R_m.$$
Moreover, let
$$\mathbf{x}_p = x_p t^p + x_{p+1} t^{p+1} + \cdots + x_m t^m,$$
$$\mathbf{y}_q = y_q t^q + y_{q+1} t^{q+1} + \cdots + y_m t^m, \text{ and}$$
$$\mathbf{z}_r = z_r t^r + z_{r+1} t^{r+1} + \cdots + z_m t^m.$$
(For $p > m$, $q > m$ or $r > m$, we think of the right-hand sides as 0.) For a polynomial $f \in k[x, y, z]$, we expand $f(\mathbf{x}_p, \mathbf{y}_q, \mathbf{z}_r)$ as
$$f(\mathbf{x}_p, \mathbf{y}_q, \mathbf{z}_r) = f_{pqr}^{(0)} + f_{pqr}^{(1)} t + \cdots + f_{pqr}^{(m)} t^m$$
in $R_m[t]/\langle t^{m+1} \rangle$, where $f_{pqr}^{(j)} \in R_m$. Clearly, $f_{pqr}^{(j)} \in k[x_p, ..., x_m, y_q, ..., y_m, z_r, ..., z_m]$ holds.

Remark 2.6. For $0 \leq j \leq m$, we have
$$f^{(j)} \equiv f_{pqr}^{(j)} \mod L_{pqr}.$$
In particular,
$$L_{pqr} + \langle f^{(0)}, ..., f^{(m)} \rangle = L_{pqr} + \langle f_{pqr}^{(0)}, ..., f_{pqr}^{(m)} \rangle.$$

Next, we define the truncation morphisms. Let $m, m' \in \mathbb{Z}_{\geq 0}$ with $m \geq m'$.

Definition 2.7. The *truncation morphism*
$$\pi_{m,m'}^X : X_m \to X_{m'}$$
is defined as the morphism induced by the natural morphism Spec $k[t]/\langle t^{m'+1} \rangle \to$ Spec $k[t]/\langle t^{m+1} \rangle$. We write
$$\varpi_{m,m'}^X : \Gamma(X_{m'}, \mathcal{O}_{X_{m'}}) \to \Gamma(X_m, \mathcal{O}_{X_m})$$
for the corresponding ring homomorphism. When $X = \mathbb{A}^3$, we use $\pi_{m,m'}$ (resp. $\varpi_{m,m'}$) instead of $\pi_{m,m'}^{\mathbb{A}^3}$ (resp. $\varpi_{m,m'}^{\mathbb{A}^3}$). We write π_m^X (resp. ϖ_m^X) for $\pi_{m,0}^X$ (resp. $\varpi_{m,0}^X$) and regard it as a morphism from X_m to X (resp. $\Gamma(X, \mathcal{O}_X)$ to $\Gamma(X_m, \mathcal{O}_{X_m})$).

We are interested in the fiber of the truncation morphism at a singular point, so we introduce the following term.

Definition 2.8. Let $X \subseteq \mathbb{A}^3$ be a surface with an isolated singular point at the origin 0 and m a positive integer. The fiber $\pi_m^{-1}(0)$ of the truncation morphism at the singular point is called the *singular fiber* and is denoted by X_m^0.

The following remark explains the relation between ideals in $\Gamma(X_m, \mathcal{O}_{X_m})$ and $\Gamma((\mathbb{A}^3)_m, \mathcal{O}_{(\mathbb{A}^3)_m})$.

Remark 2.9. Recall that
$$R_m := \Gamma((\mathbb{A}^3)_m, \mathcal{O}_{(\mathbb{A}^3)_m}) = k[x_0, ..., x_m, y_0, ..., y_m, z_0, ..., z_m].$$
Let X be $\mathbf{V}(f)$, $i_m : X_m \to (\mathbb{A}^3)_m$ the natural inclusion and $\iota_m : R_m \to \Gamma(X_m, \mathcal{O}_{X_m})$ the corresponding ring homomorphism. For any $I \subset R_m$ with $I \supset \langle f^{(0)}, ..., f^{(m)} \rangle$, we set $\tilde{I} := \iota_m(I)$. Then, $i_m(\mathbf{V}(\tilde{I})) = \mathbf{V}(I)$ clearly holds. Under this inclusion morphism i_m, X_m and its closed subschemes are identified with closed subschemes in $(\mathbb{A}^3)_m$.

We can describe the inverse images by the truncation morphism as follows: Suppose $m \geq m'$ and let $Z \subseteq X_{m'}$ be a closed subscheme defined by $\tilde{I} \subset \Gamma(X_{m'}, \mathcal{O}_{X_{m'}})$, $I \subset R_{m'}$ an ideal with $I \supset \langle f^{(0)}, ..., f^{(m')} \rangle$ and $\iota_{m'}(I) = \tilde{I}$. Then a defining ideal of $(\pi_{m,m'}^X)^{-1}(Z) = (\pi_{m,m'}^X)^{-1}(\mathbf{V}(I))$ is $\varpi_{m,m'}^X(\tilde{I}) \cdot \Gamma(X_m, \mathcal{O}_{X_m})$ and this ideal satisfies
$$\varpi_{m,m'}^X(\tilde{I}) \cdot \Gamma(X_m, \mathcal{O}_{X_m}) = \iota_m(\varpi_{m,m'}(I) \cdot R_m + \langle f^{(m'+1)}, f^{(m'+2)}, ..., f^{(m)} \rangle).$$

Hence, under the above identification, the defining ideal of $(\pi^X_{m,m'})^{-1}(Z) \subseteq (\mathbb{A}^3)_m$ is

$$\varpi_{m,m'}(I) \cdot R_m + \langle f^{(m'+1)}, f^{(m'+2)}, ..., f^{(m)} \rangle$$

in R_m.

Throughout this paper, we identify X_m and its closed subschemes with the corresponding subschemes in $(\mathbb{A}^3)_m$ by i_m.

The following proposition and remark describe the inverse images of the smooth locus by the truncation morphism.

Proposition 2.10 ([2, Proposition 2.4])**.** *Let X, Y be schemes over k. If $f : X \to Y$ is an étale morphism, then $X_m \cong Y_m \times_Y X$ for every $m \in \mathbb{Z}_{\geq 0}$.*

Remark 2.11. If X is an n-dimensional variety, then $\pi_m : (X_{\mathrm{sm}})_m \to X_{\mathrm{sm}}$ is a locally trivial fibration with the fiber $\mathbb{A}^{nm}$ by Proposition 2.10, where $X_{\mathrm{sm}} = X - \mathrm{Sing}\, X$. Hence $\overline{\pi_m^{-1}(X_{\mathrm{sm}})}$ is an irreducible component of X_m, and we call it the main component.

In general, the m-th jet scheme X_m is not irreducible. However, if X has only rational double points, then X_m is irreducible and consists of the main component ([4, Corollary 10.2.9]).

Next, we recall the defining equations of singularities in $\mathbb{A}^3$ whose resolution graphs are of type D_4. In positive characteristic other than 2, a surface singularity of type D_4 can be defined by $x^2 - y^2z + z^3$, which is the same as in the case of characteristic 0. On the other hand, in characteristic 2, there are two singularities ([1, Section 3]): a singularity of type D_4^0, defined by

$$f = x^2 + y^2z + yz^2,$$

and a singularity of type D_4^1, defined by

$$g = x^2 + y^2z + yz^2 + xyz.$$

In characteristic different from 2, they both give a singularity of type D_4.

To conclude this section, we give the lemmas that will be used in the following sections.

One key point in the description of the singular fibers in [7] was that $f^{(j)}$ is often of the form $Ay_i + B$ (resp. $Az_i + B$) where A and B do not contain y_l (resp. z_l) with $l \geq i$. Hence we set up the following notation.

Notation 2.12. Let $h \in R_m$. The sum of the terms in h containing y_i (resp. z_i) with the largest index i is denoted by $T_y(h)$ (resp. $T_z(h)$).

Example 2.13. If $h = x_3^2 + y_2^2z_2 + y_2z_2^2$, then $T_y(h) = y_2^2z_2 + y_2z_2^2$. If $h = x_3^2 + y_2^2z_2 + y_2z_2^2 + y_4z_1^2$, then $T_y(h) = y_4z_1^2$.

For any polynomials f and g, we have the following.

Lemma 2.14. *Let k be a field of characteristic 2 and $f, g \in k[x, y, z]$ be defined by $f = x^2 + y^2z + yz^2$ and $g = x^2 + y^2z + yz^2 + xyz$. Assume $p, q, r \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}_{\geq 0}$ with $l \leq m$. Then we have*

$$f^{(l)}_{pqr} = \sum_{u \geq p, 2u = l} x_u^2 + \sum_{v \geq q, w \geq r, 2v+w=l} y_v^2 z_w + \sum_{v \geq q, w \geq r, v+2w=l} y_v z_w^2$$

and

$$g^{(l)}_{pqr} = \sum_{u \geq p, 2u = l} x_u^2 + \sum_{v \geq q, w \geq r, 2v+w=l} y_v^2 z_w + \sum_{v \geq q, w \geq r, v+2w=l} y_v z_w^2 + \sum_{u \geq p, v \geq q, w \geq r, u+v+w=l} x_u y_v z_w.$$

Here, if there are no u, v and w satisfying the conditions, we regard the sums as 0. Furthermore, the following hold.

(a) *If $l < 2p$, $l < 2q + r$ and $l < q + 2r$, then we have*

$$f^{(l)}_{pqr} = g^{(l)}_{pqr} = 0.$$

(b) *If $l = 2p$, $l < 2q + r$ and $l < q + 2r$, then we have*

$$f^{(l)}_{pqr} = g^{(l)}_{pqr} = x_p^2.$$

(c) *If* $(p >)q = r$ *and* $l = 2p = 3q$, *then*

$$f^{(l)}_{pqq} = g^{(l)}_{pqq} = x_p^2 + y_q^2 z_q + y_q z_q^2.$$

(d) *If* $(p >)q = r$ *and* $l = 3q < 2p$, *then*

$$f^{(l)}_{pqq} = g^{(l)}_{pqq} = y_q^2 z_q + y_q z_q^2 = y_q z_q (y_q + z_q).$$

(e) *If* $p \geq q > r$ *and* $l \geq 2p = q + 2r$, *then*

$$T_y(f^{(l)}_{pqr}) = T_y(g^{(l)}_{pqr}) = y_{l-2r} z_r^2.$$

(f) *If* $p \geq r > q$ *and* $l \geq 2p = 2q + r$, *then*

$$T_z(f^{(l)}_{pqr}) = T_z(g^{(l)}_{pqr}) = y_q^2 z_{l-2q}.$$

(g) *If* $p > q = r$ *and* $l > 3q$, *then*

$$T_y(f^{(l)}_{pqq}) = T_y(g^{(l)}_{pqq}) = y_{l-2q} z_q^2.$$

(h) *If* $p > q = r$ *and* $l > 3q$, *then*

$$T_z(f^{(l)}_{pqq}) = T_z(g^{(l)}_{pqq}) = y_q^2 z_{l-2q}.$$

Remark 2.15. We note that if p, q, r satisfy the above conditions (a)–(d), then there are no terms coming from xyz in $g^{(l)}_{pqr}$. In particular, the polynomial $g^{(2p)}_{pqq} = x_p^2 + y_q^2 z_q + y_q z_q^2$ appearing in (c) has the same form as the defining equation f of a singular surface of type D_4^0. Since the D_4^0-type singular surface is irreducible, $g^{(2p)}_{pqq}$ is also irreducible in $k[x_p, y_q, z_q]$. Moreover, if p, q, r satisfy the above conditions (e)–(h), then there are no terms coming from xyz in $T_y(g^{(l)}_{pqr})$ and $T_z(g^{(l)}_{pqr})$.

Proof. First of all, we note that

$$\begin{aligned}\mathbf{x}_p^2 &= (x_p t^p + x_{p+1} t^{p+1} + \cdots + x_m t^m)^2 \\ &= x_p^2 t^{2p} + x_{p+1}^2 t^{2(p+1)} + \cdots + x_m^2 t^{2m},\end{aligned}$$

and similarly for $\mathbf{y}_q^2$ and $\mathbf{z}_r^2$, since we are working in characteristic 2.

Since $f^{(l)}_{pqr}$ and $g^{(l)}_{pqr}$ are the coefficient of t^l in the expansion of $f(\mathbf{x}_p, \mathbf{y}_q, \mathbf{z}_r) = \mathbf{x}_p^2 + \mathbf{y}_q^2 \mathbf{z}_r + \mathbf{y}_q \mathbf{z}_r^2$ and $g(\mathbf{x}_p, \mathbf{y}_q, \mathbf{z}_r) = \mathbf{x}_p^2 + \mathbf{y}_q^2 \mathbf{z}_r + \mathbf{y}_q \mathbf{z}_r^2 + \mathbf{x}_p \mathbf{y}_q \mathbf{z}_r$, we obtain

$$f^{(l)}_{pqr} = \sum_{u \geq p, 2u = l} x_u^2 + \sum_{v \geq q, w \geq r, 2v + w = l} y_v^2 z_w + \sum_{v \geq q, w \geq r, v + 2w = l} y_v z_w^2$$

and

$$g^{(l)}_{pqr} = \sum_{u \geq p, 2u = l} x_u^2 + \sum_{v \geq q, w \geq r, 2v + w = l} y_v^2 z_w + \sum_{v \geq q, w \geq r, v + 2w = l} y_v z_w^2 + \sum_{u \geq p, v \geq q, w \geq r, u + v + w = l} x_u y_v z_w$$

by a direct calculation.

Before we begin the proof of (a)–(h), we note that the difference between $f^{(l)}$ and $g^{(l)}$ is the terms coming from xyz. As we saw in Remark 2.15, the results of Lemma 2.14 are the same for f and g. Noting that every term in $(xyz)^{(l)}$ is of the form $x_u^i y_v^j z_w^k$ with $i, j, k > 0$ and that $f^{(l)}$ contains no such terms, it suffices to prove Lemma 2.14 for g.

Let us prove (a), (b), (c) and (d). Note, first of all, that the lowest exponent of t appearing in $\mathbf{x}_p^2$ (resp. $\mathbf{y}_q^2 \mathbf{z}_r$, $\mathbf{y}_q \mathbf{z}_r^2$, $\mathbf{x}_p \mathbf{y}_q \mathbf{z}_r$) is $2p$ (resp. $2q + r$, $q + 2r$, $p + q + r$).

For (a), we only have to show that the lowest exponent of t in $g(\mathbf{x}_p, \mathbf{y}_q, \mathbf{z}_r)$ is greater than l. By the conditions in (a), we have $l < 2p$, $l < 2q + r$ and $l < q + 2r$, so we only have to check $l < p + q + r$. From the assumption, we have $p > l/2$ and $q + r > 2l/3$, and then that $p + q + r > 7l/6 > l$. Thus $g^{(l)}_{pqr} = 0$ holds.

For (b), again we only have to show that $p + q + r > l$. By assumption in (b) $p = l/2$ and $q + r > 2l/3$, so $p + q + r > 7l/6 > l$.

For (c), we have $l = 2p = 3q$ by assumption and $\mathbf{x}_p^2$, $\mathbf{y}_q^2\mathbf{z}_q$ and $\mathbf{y}_q\mathbf{z}_q^2$ give rise to terms x_p^2, $y_q^2 z_q$ and $y_q z_q^2$, so we only have to show that $p + q + q > l$. From the assumption $p > q$, so $l = 3q < p + q + q$. Hence $g_{pqq}^{(l)} = x_p^2 + y_q^2 z_q + y_q z_q^2$.

For (d), we have $2p > l$ and $p > q$ by assumption, so $p + q + q > 3q = l$. Therefore,

$$g_{pqq}^{(3q)} = y_q^2 z_q + y_q z_q^2 = y_q z_q (y_q + z_q)$$

holds.

Before proving the statements (e)–(h), let us find out the terms containing y_i (resp. z_i) with the largest i in $(\mathbf{y}_q^2\mathbf{z}_r)^{(l)}$, $(\mathbf{y}_q\mathbf{z}_r^2)^{(l)}$ and $(\mathbf{x}_p\mathbf{y}_q\mathbf{z}_r)^{(l)}$, where we write the coefficient of t^l in $\mathbf{y}_q^2\mathbf{z}_r$ as $(\mathbf{y}_q^2\mathbf{z}_r)^{(l)}$, and so on.

First, we look at $T_y((\mathbf{y}_q^2\mathbf{z}_r)^{(l)})$ (resp. $T_z((\mathbf{y}_q\mathbf{z}_r^2)^{(l)})$) (see Notation 2.12). If $l < 2q + r$ (resp. $l < q + 2r$), it is obvious that $(\mathbf{y}_q^2\mathbf{z}_r)^{(l)} = 0$ (resp. $(\mathbf{y}_q\mathbf{z}_r^2)^{(l)} = 0$), so we assume $l \geq 2q + r$ (resp. $l \geq q + 2r$). If $l - r$ (resp. $l - q$) is even, then

$$T_y((\mathbf{y}_q^2\mathbf{z}_r)^{(l)}) = y_{\frac{l-r}{2}}^2 z_r \ \ (\text{resp. } T_z((\mathbf{y}_q\mathbf{z}_r^2)^{(l)}) = y_q z_{\frac{l-q}{2}}^2),$$

and if $l - r$ (resp. $l - q$) is odd, then

$$T_y((\mathbf{y}_q^2\mathbf{z}_r)^{(l)}) = y_{\frac{l-r-1}{2}}^2 z_{r+1} \ \ (\text{resp. } T_z((\mathbf{y}_q\mathbf{z}_r^2)^{(l)}) = y_{q+1} z_{\frac{l-q-1}{2}}^2).$$

Second, assuming $l \geq q + 2r$ (resp. $l \geq 2q + r$),

$$T_y((\mathbf{y}_q\mathbf{z}_r^2)^{(l)}) = y_{l-2r} z_r^2 \ \ (\text{resp. } T_z((\mathbf{y}_q^2\mathbf{z}_r)^{(l)}) = y_q^2 z_{l-2q}).$$

Third, if $l \geq p + q + r$, then

$$T_y((\mathbf{x}_p\mathbf{y}_q\mathbf{z}_r)^{(l)}) = x_p y_{l-p-r} z_r \ \ (\text{resp. } T_z((\mathbf{x}_p\mathbf{y}_q\mathbf{z}_r)^{(l)}) = x_p y_q z_{l-p-q}),$$

and if $l < p + q + r$, then $(\mathbf{x}_p\mathbf{y}_q\mathbf{z}_r)^{(l)} = 0$.

Now, we prove (e) (resp. (g)). Since the terms $T_y((\mathbf{y}_q^2\mathbf{z}_r)^{(l)})$, $T_y((\mathbf{y}_q\mathbf{z}_r^2)^{(l)})$ and $T_y((\mathbf{x}_p\mathbf{y}_q\mathbf{z}_r)^{(l)})$ are as above, we only have to show that $l - 2r > (l - r)/2$ and $l - 2r > l - p - r$. First, we prove $l - 2r > (l - r)/2$. By the assumption $q > r$ and $l \geq q + 2r$ (resp. $l > 3q = 3r$), so $l > 3r$ and this is equivalent to $l - 2r > (l - r)/2$. Moreover, $p > r$ by the assumption. So $l - 2r > l - p - r$. Thus, $T_y(g_{pqr}^{(l)}) = y_{l-2r} z_r^2$ (resp. $T_y(g_{pqq}^{(l)}) = y_{l-2q} z_q^2 = y_{l-2r} z_r^2$).

By symmetry, we also have (f) and (h). □

Focusing on g, we have the following lemma.

Lemma 2.16. *Assume $p, q, r \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}_{\geq 0}$ with $l \leq m$.*

(a,g) *If $l \notin 2\mathbb{Z}$ and $l < 2q + r$, $l < q + 2r$ and $l < p + q + r$, then we have*

$$g_{pqr}^{(l)} = 0.$$

(b,g) *If $l = 2p'$ with $p \leq p'$, $l < 2q + r$, $l < q + 2r$ and $l < p + q + r$, then we have*

$$g_{pqr}^{(l)} = x_{p'}^2.$$

The proof is basically the same as that of Lemma 2.14(a) and (b).

The next lemma is one of the key points in proving the irreducibility of certain closed subsets of the singular fiber.

Lemma 2.17. *Let S be the surface defined by f or g, S_m the m-th jet scheme of S and S_m^0 the singular fiber for $m \geq 1$.*

(a) *If $Z \subseteq S_m$ is an irreducible component, then* $\operatorname{codim}_{(\mathbb{A}^3)_m} Z \leq m+1$ *(or equivaelntly* $\dim Z \geq 2m+2$*).*
(b) *If $Z \subseteq S_m^0$ is an irreducible component, then* $\operatorname{codim}_{(\mathbb{A}^3)_m} Z \leq m+2$ *(or equivaelntly* $\dim Z \geq 2m+1$*).*

Proof. (a) The m-th jet scheme S_m is defined by the ideal generated by $m + 1$ elements $\{f^{(0)}, ..., f^{(m)}\}$, so for any irreducible component $Z \subseteq S_m$,

$$\operatorname{codim}_{(\mathbb{A}^3)_m} Z \leq m+1.$$

by Krull's height theorem.
(b) The singular fiber S_m^0 is defined by

$$L_{111} + \langle f^{(0)}, ..., f^{(m)} \rangle = L_{111} + \langle f_{111}^{(0)}, ..., f_{111}^{(m)} \rangle$$
$$(\text{resp. } L_{111} + \langle g^{(0)}, ..., g^{(m)} \rangle = L_{111} + \langle g_{111}^{(0)}, ..., g_{111}^{(m)} \rangle),$$

from Remark 2.9 and Remark 2.6. By Lemma 2.14(a), we have

$$f_{111}^{(0)} = f_{111}^{(1)} = 0 \text{ (resp. } g_{111}^{(0)} = g_{111}^{(1)} = 0)$$

and S_m^0 is defined by the ideal generated by $3 + m + 1 - 2 = m + 2$ elements. Thus as in (a), for any irreducible component $Z \subseteq S_m^0$,

$$\operatorname{codim}_{(\mathbb{A}^3)_m} Z \leq m+2.$$

□

3. Jet schemes of a singular surface of type D_4^0

In this section, we deal with a singular surface of type D_4^0. First, we find the irreducible decomposition of the singular fiber. This can be done by the method of Mourtada [7]. Next, we determine the inclusion relations among intersections of irreducible components of the singular fiber.

For the determination of the irreducible decomposition in the D_4^0 case, the arguments are almost the same as in characteristic 0.

Remark 3.1. In positive characteristic not equal to 2, a singular surface of type D_4 is defined by $h = x^2 - y^2 z + z^3$ in $\mathbb{A}^3$. We consider the transformation

$$\begin{cases} x \mapsto x, \\ y \mapsto \frac{1}{\sqrt[6]{4}} y + \frac{\sqrt[3]{4}}{2} z, \\ z \mapsto -\frac{1}{\sqrt[6]{4}} y + \frac{\sqrt[3]{4}}{2} z. \end{cases}$$

Then the polynomial $f = x^2 + y^2 z + yz^2$ is mapped to h.

Let $S = \mathbf{V}(f) \subseteq \mathbb{A}^3$ be a singular surface of type D_4^0 over an algebraically closed field k of characteristic 2, S_m the m-th jet scheme of S, S_m^0 the singular fiber of S_m and $R_m = k[x_0, ..., x_m, y_0, ..., y_m, z_0, ..., z_m]$. Moreover, we set

$$L_{pqr} = \langle x_0, ..., x_{p-1}, y_0, ..., y_{q-1}, z_0, ..., z_{r-1} \rangle$$

for positive integers $p, q, r \in \mathbb{Z}_{>0}$ with $p, q, r \leq m+1$.

Definition 3.2. For $m \geq 1$, we define the following ideals in R_m:

$$\begin{aligned} J_m^1 &= L_{211} + \langle y_1 \rangle \quad + \langle f^{(0)}, ..., f^{(m)} \rangle = L_{221} + \langle f^{(0)}, ..., f^{(m)} \rangle, \\ J_m^2 &= L_{211} + \langle z_1 \rangle \quad + \langle f^{(0)}, ..., f^{(m)} \rangle = L_{212} + \langle f^{(0)}, ..., f^{(m)} \rangle, \\ J_m^3 &= L_{211} + \langle y_1 + z_1 \rangle + \langle f^{(0)}, ..., f^{(m)} \rangle. \end{aligned}$$

By Remark 2.9 and $J_m^i \supset L_{111}$, these ideals include the defining ideal of the singular fiber, and hence correspond to closed subsets in the singular fiber S_m^0.

We have the following symmetries:

Notation 3.3. Let φ_1 and φ_2 be the automorphisms of R_m defined by

$$\varphi_1 : \begin{cases} x_i \mapsto x_i, \\ y_i \mapsto z_i, \\ z_i \mapsto y_i, \end{cases} \quad \varphi_2 : \begin{cases} x_i \mapsto x_i, \\ y_i \mapsto y_i, \\ z_i \mapsto y_i + z_i. \end{cases}$$

These automorphisms φ_1 and φ_2 induce ring isomorphisms

$$(\varphi_1)_{z_1} : (R_m)_{z_1} \to (R_m)_{y_1} \text{ and } (\varphi_2)_{y_1} : (R_m)_{y_1} \to (R_m)_{y_1}.$$

We write the isomorphisms corresponding to φ_1, φ_2, $(\varphi_1)_{z_1}$ and $(\varphi_2)_{y_1}$ as ψ_1, ψ_2, $(\psi_1)_{z_1}$ and $(\psi_2)_{y_1}$, respectively. For simplicity, we write $(\varphi_1)_{z_1}$, $(\varphi_2)_{y_1}$, $(\psi_1)_{z_1}$ and $(\psi_2)_{y_1}$ as φ_1, φ_2, ψ_1 and ψ_2.

Lemma 3.4. (a) *For $m \geq 1$, $i \in \{0, ..., m\}$ and $k = 1, 2$, φ_k preserve $f^{(i)}$ i.e.,*

$$\varphi_k(f^{(i)}) = f^{(i)}$$

in R_m. In particular, the morphisms ψ_k preserve S_m.

(b) *For $m \geq 1$,*

$$\varphi_1(J_m^1 \cdot (R_m)_{z_1}) = J_m^2 \cdot (R_m)_{y_1}$$

and

$$\varphi_2(J_m^2 \cdot (R_m)_{y_1}) = J_m^3 \cdot (R_m)_{y_1}.$$

(c) *The morphisms φ_1, φ_2, ψ_1 and ψ_2 preserve the union, the intersection and the inclusion relations of sets.*

Proof. (a) Note that the automorphisms φ_1 and φ_2 are induced by the automorphisms of $k[x, y, z]$ defined by

$$\overline{\varphi_1} : \begin{cases} x \mapsto x, \\ y \mapsto z, \\ z \mapsto y, \end{cases} \quad \overline{\varphi_2} : \begin{cases} x \mapsto x, \\ y \mapsto y, \\ z \mapsto y + z, \end{cases}$$

and $\overline{\varphi_1}(f) = x^2 + z^2 y + zy^2 = f$ and

$$\begin{aligned} \overline{\varphi_2}(f) &= x^2 + y^2(y + z) + y(y + z)^2 \\ &= x^2 + y(y + z)(y + y + z) \\ &= x^2 + y(y + z)z \qquad = x^2 + y^2 z + yz^2, \end{aligned}$$

where $y + y = 0$ since k is a field of characteristic 2. Since

$$\begin{aligned} f\left(\sum_{i=0}^m x_i t^i, \sum_{i=0}^m y_i t^i, \sum_{i=0}^m z_i t^i\right) &= \overline{\varphi_k}(f)\left(\sum_{i=0}^m x_i t^i, \sum_{i=0}^m y_i t^i, \sum_{i=0}^m z_i t^i\right) \\ &= f\left(\sum_{i=0}^m \varphi_k(x_i) t^i, \sum_{i=0}^m \varphi_k(y_i) t^i, \sum_{i=0}^m \varphi_k(z_i) t^i\right) \\ &= \sum_{i=0}^m \varphi_k(f^{(i)}) t^i \end{aligned}$$

in $R_m[t]/\langle t^{m+1}\rangle$ for $k = 1, 2$, φ_k preserves the polynomials $f^{(i)}$ ($i \in \{0, ..., m\}$), and the induced automorphism ψ_k preserves S_m.

(b) We can easily check that

$$\varphi_1((L_{211} + \langle y_1\rangle) \cdot (R_m)_{z_1}) = (L_{211} + \langle z_1\rangle) \cdot (R_m)_{y_1}$$

and

$$\varphi_2((L_{211} + \langle z_1\rangle) \cdot (R_m)_{y_1}) = (L_{211} + \langle y_1 + z_1\rangle) \cdot (R_m)_{y_1}.$$

By the assertion (a), we have $\varphi_1(J_m^1 \cdot (R_m)_{z_1}) = J_m^2 \cdot (R_m)_{y_1}$ and $\varphi_2(J_m^2 \cdot (R_m)_{y_1}) = J_m^3 \cdot (R_m)_{y_1}$.

(c) The morphisms φ_1, φ_2, ψ_1 and ψ_2 are isomorphisms, so under these morphisms, the union, the intersection and the inclusion relations of sets are preserved. □

We will show that J_m^1, J_m^2 and J_m^3 define irreducible components of certain open subsets of S_m^0.

Proposition 3.5. *For $m \geq 3$, the ideal $J_m^1 \cdot (R_m)_{z_1}$ is a prime ideal in $(R_m)_{z_1}$, and $J_m^2 \cdot (R_m)_{y_1}$ and $J_m^3 \cdot (R_m)_{y_1}$ are prime ideals in $(R_m)_{y_1}$. Moreover, the heights of the ideals $J_m^1 \cdot (R_m)_{z_1}$, $J_m^2 \cdot (R_m)_{y_1}$ and $J_m^3 \cdot (R_m)_{y_1}$ are $m + 2$.*

Proof. We prove that the ideal $J_m^1 \cdot (R_m)_{z_1}$ is prime of height $m + 2$. Then by Lemma 3.4(b), it follows that $J_m^2 \cdot (R_m)_{y_1}$ and $J_m^3 \cdot (R_m)_{y_1}$ are prime ideals of height $m + 2$ in $(R_m)_{y_1}$.

First, we note that

$$J_m^1 = L_{221} + \langle f^{(0)}, ..., f^{(m)} \rangle = L_{221} + \langle f_{221}^{(0)}, ..., f_{221}^{(m)} \rangle$$

by Remark 2.6. Then, we have

$$f_{221}^{(0)} = f_{221}^{(1)} = f_{221}^{(2)} = f_{221}^{(3)} = 0$$

by Lemma 2.14(a) and

$$T_y(f_{221}^{(l)}) = y_{l-2} z_1^2$$

for $4 \leq l \leq m$ by Lemma 2.14(e). Hence there exists $h_l \in k[x_2, ..., x_m, y_2, ..., y_{l-3}, z_1, ..., z_m]$ such that

$$f_{221}^{(l)} = y_{l-2} z_1^2 + h_l$$

for $4 \leq l \leq m$. Since

$$y_{l-2} + \frac{h_l}{z_1^2} = \frac{1}{z_1^2} f_{221}^{(l)} \in J_m^1 \cdot (R_m)_{z_1},$$

we have

$$\begin{aligned} J_m^1 \cdot (R_m)_{z_1} &= (L_{221} + \langle f_{221}^{(0)}, ..., f_{211}^{(m)} \rangle) \cdot (R_m)_{z_1} \\ &= \left(L_{221} + \left\langle y_2 + \frac{h_4}{z_1^2}, ..., y_{m-2} + \frac{h_m}{z_1^2} \right\rangle \right) \cdot (R_m)_{z_1}. \end{aligned}$$

Thus the ideal $J_m^1 \cdot (R_m)_{z_1}$ is a prime ideal of $(R_m)_{z_1}$ and the height of $J_m^1 \cdot (R_m)_{z_1}$ is $m + 2$. □

Let us define some ideals of R_m and the corresponding closed subsets of $(\mathbb{A}^3)_m$.

Definition 3.6. For $m \geq 1$, we define

$$\begin{aligned} I_m^0 &= L_{222} + \langle f^{(0)}, ..., f^{(m)} \rangle, \\ I_m^1 &= J_m^1 \cdot (R_m)_{z_1} \cap R_m, \\ I_m^2 &= J_m^2 \cdot (R_m)_{y_1} \cap R_m, \\ I_m^3 &= J_m^3 \cdot (R_m)_{y_1} \cap R_m. \end{aligned}$$

Furthermore, we define closed subsets

$$Z_m^i := \mathbf{V}(I_m^i)$$

for $0 \leq i \leq 3$.

Remark 3.7. (a) By Proposition 3.5, the ideals I_m^1, I_m^2 and I_m^3 for $m \geq 3$ are prime of height $m + 2$, and the closed subsets Z_m^1, Z_m^2 and Z_m^3 are irreducible of codimension $m + 2$ in $(\mathbb{A}^3)_m$.

(b) By Definition 3.6, we have

$$Z_m^1 = \overline{\mathbf{V}(J_m^1) \cap \mathbf{D}(z_1)},$$
$$Z_m^2 = \overline{\mathbf{V}(J_m^2) \cap \mathbf{D}(y_1)},$$
$$Z_m^3 = \overline{\mathbf{V}(J_m^3) \cap \mathbf{D}(y_1)},$$

where $\mathbf{D}(h)$ ($h \in R_m$) is an open subset defined by $(\mathbb{A}^3)_m - \mathbf{V}(\langle h \rangle)$. Moreover, we have $y_1 + z_1 \in J_m^3$, hence we have $I_m^3 = J_m^3 \cdot (R_m)_{y_1} \cap R_m = J_m^3 \cdot (R_m)_{z_1} \cap R_m$ and therefore

$$Z_m^3 = \overline{\mathbf{V}(J_m^3) \cap \mathbf{D}(z_1)}.$$

(c) For $m \geq 4$, we have $f_{222}^{(4)} = x_2^2$ by Lemma 2.14(b). Hence

$$Z_m^0 = \mathbf{V}(L_{322} + \langle f^{(0)}, ..., f^{(m)} \rangle).$$

Here, we note that Z_m^1, Z_m^2 and Z_m^3 have the following symmetries.

Lemma 3.8. *Assume $m \geq 3$. The closed subsets Z_m^0, Z_m^1, Z_m^2 and Z_m^3 are permutated by ψ_1 and ψ_2 (see Notation 3.3) as follows.*

(a) $\psi_1(Z_m^0) = Z_m^0, \psi_1(Z_m^1) = Z_m^2, \psi_1(Z_m^2) = Z_m^1$ *and* $\psi_1(Z_m^3) = Z_m^3$.
(b) $\psi_2(Z_m^0) = Z_m^0, \psi_2(Z_m^1) = Z_m^1, \psi_2(Z_m^2) = Z_m^3$ *and* $\psi_2(Z_m^3) = Z_m^2$.

Proof. (a) We may think of φ_1 as an automorphism of the quotient field of R_m which preserves R_m and maps $(R_m)_{z_1}$ to $(R_m)_{y_1}$ and $(R_m)_{y_1}$ to $(R_m)_{z_1}$. Arguing as in Lemma 3.4(b) and (c), we have

$$\varphi_1(I_m^0) = I_m^0,$$
$$\varphi_1(I_m^1) = \varphi_1(J_m^1 \cdot (R_m)_{z_1} \cap R_m) = \varphi_1(J_m^1 \cdot (R_m)_{z_1}) \cap R_m = J_m^2 \cdot (R_m)_{y_1} \cap R_m = I_m^2,$$
$$\varphi_1(I_m^2) = \varphi_1(J_m^2 \cdot (R_m)_{y_1} \cap R_m) = \varphi_1(J_m^2 \cdot (R_m)_{y_1}) \cap R_m = J_m^1 \cdot (R_m)_{z_1} \cap R_m = I_m^1,$$
$$\varphi_1(I_m^3) = \varphi_1(J_m^3 \cdot (R_m)_{z_1} \cap R_m) = \varphi_1(J_m^3 \cdot (R_m)_{z_1}) \cap R_m = J_m^3 \cdot (R_m)_{y_1} \cap R_m = I_m^3.$$

(see Remark 3.7(b)). Hence we have

$$\psi_1(Z_m^0) = Z_m^0, \psi_1(Z_m^1) = Z_m^2, \psi_1(Z_m^2) = Z_m^1 \text{ and } \psi_1(Z_m^3) = Z_m^3.$$

We can prove (b) in the same way as (a). □

In the following, we give the irreducible decomposition of the singular fiber S_m^0. First of all, we give the decomposition of S_m^0.

Proposition 3.9. *For $m \geq 3$, we have*

$$S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3.$$

Moreover, Z_m^1, Z_m^2 and Z_m^3 are pairwise distinct.

Proof. By Lemma 2.14(a) and (b), we have

$$f_{111}^{(0)} = f_{111}^{(1)} = 0,$$
$$f_{111}^{(2)} = x_1^2.$$

Hence the defining ideal of S_m^0 is

$$\sqrt{\langle x_0, y_0, z_0, x_1^2, f^{(3)}, ..., f^{(m)} \rangle} = \sqrt{\langle x_0, y_0, z_0, x_1, f_{211}^{(3)}, ..., f_{211}^{(m)} \rangle}.$$

Using Lemma 2.14(d), we have

$$f_{211}^{(3)} = y_1 z_1 (y_1 + z_1).$$

Thus we have

$$\begin{aligned} S_m^0 =&\mathbf{V}(\langle x_0, x_1, y_0, y_1, z_0, f^{(4)}, ..., f^{(m)}\rangle) \cup \mathbf{V}(\langle x_0, x_1, y_0, z_0, z_1, f^{(4)}, ..., f^{(m)}\rangle) \\ &\cup \mathbf{V}(\langle x_0, x_1, y_0, z_0, y_1 + z_1, f^{(4)}, ..., f^{(m)}\rangle) \\ =&\mathbf{V}(J_m^1) \cup \mathbf{V}(J_m^2) \cup \mathbf{V}(J_m^3). \end{aligned}$$

Since $(\mathbb{A}^3)_m = \mathbf{D}(y_1) \cup \mathbf{D}(z_1) \cup \mathbf{V}(y_1, z_1)$ and S_m^0 is closed, we have

$$S_m^0 = S_m^0 \cap (\mathbb{A}^3)_m = \overline{S_m^0 \cap \mathbf{D}(y_1)} \cup \overline{S_m^0 \cap \mathbf{D}(z_1)} \cup (S_m^0 \cap \mathbf{V}(y_1, z_1)).$$

Note that $\mathbf{V}(J_m^1) \cap \mathbf{D}(y_1) = \emptyset$ since $y_1 \in J_m^1$. Thus

$$S_m^0 \cap \mathbf{D}(y_1) = (\mathbf{V}(J_m^2) \cap \mathbf{D}(y_1)) \cup (\mathbf{V}(J_m^3) \cap \mathbf{D}(y_1)).$$

Similarly,

$$S_m^0 \cap \mathbf{D}(z_1) = (\mathbf{V}(J_m^1) \cap \mathbf{D}(z_1)) \cup (\mathbf{V}(J_m^3) \cap \mathbf{D}(z_1)).$$

By Remark 3.7, we have

$$\overline{S_m^0 \cap \mathbf{D}(y_1)} = Z_m^2 \cup Z_m^3$$

and

$$\overline{S_m^0 \cap \mathbf{D}(z_1)} = Z_m^1 \cup Z_m^3.$$

Clearly, we have $S_m^0 \cap \mathbf{V}(y_1, z_1) = \mathbf{V}(I_m^0) = Z_m^0$. Therefore,

$$\begin{aligned} S_m^0 &= \overline{\mathbf{V}(J_m^1) \cap \mathbf{D}(z_1)} \cup \overline{\mathbf{V}(J_m^2) \cap \mathbf{D}(y_1)} \cup \overline{\mathbf{V}(J_m^3) \cap \mathbf{D}(y_1)} \cup (S_m^0 \cap \mathbf{V}(y_1, z_1)) \\ &= Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3. \end{aligned}$$

Finally, we check that Z_m^1, Z_m^2 and Z_m^3 are pairwise distinct. By Lemma 3.8, it suffices to show that $Z_m^3 \not\subseteq Z_m^1$. The jet $P = (0, t, t)$ corresponds to the point where $y_1 = z_1 = 1$ and all of the other coordinates are 0, and we see $P \in Z_m^3$ and $P \notin \mathbf{V}(J_m^1) \supseteq Z_m^1$. Hence $Z_m^3 \not\subseteq Z_m^1$. □

Next, we give the irreducible decomposition of the singular fiber for small m.

Proposition 3.10. *For $0 \le m \le 4$, the irreducible decomposition of the singular fiber S_m^0 is as follows.*

(a) $S_0^0 = \mathbf{V}(L_{111})$,
(b) $S_1^0 = \mathbf{V}(L_{111})$,
(c) $S_2^0 = \mathbf{V}(L_{211})$,
(d) $S_3^0 = Z_3^1 \cup Z_3^2 \cup Z_3^3$,
(e) $S_4^0 = Z_4^1 \cup Z_4^2 \cup Z_4^3$.

Moreover, for $1 \le m \le 4$, the codimension of any irreducible component $Z \subseteq S_m^0$ is

$$\operatorname{codim}_{(\mathbb{A}^3)_m} Z = m + 2.$$

Proof. (a) By Lemma 2.14(a), $f_{111}^{(0)} = 0$, so $S_0^0 = \mathbf{V}(L_{111})$.

(b) By Lemma 2.14(a), $f_{111}^{(1)} = 0$, so $S_1^0 = \mathbf{V}(L_{111})$.

(c) By Lemma 2.14(b), $f_{111}^{(2)} = x_1^2$, so $S_2^0 = \mathbf{V}(L_{211})$.

(d) By Proposition 3.9 and Remark 3.7(a), $S_3^0 = Z_3^0 \cup Z_3^1 \cup Z_3^2 \cup Z_3^3$ and Z_3^1, Z_3^2 and Z_3^3 are irreducible and pariwise distinct. Note that $J_3^1 = L_{221}$ is prime, so $I_3^1 = J_3^1 \subsetneq L_{222} = I_3^0$. Hence $Z_3^1 \supsetneq Z_3^0$. Thus the irreducible decomposition of $S_3^0 = Z_3^1 \cup Z_3^2 \cup Z_3^3$.

(e) By Proposition 3.9,

$$S_4^0 = Z_4^0 \cup Z_4^1 \cup Z_4^2 \cup Z_4^3.$$

Here, $Z_4^0 = S_4^0 \cap \mathbf{V}(y_1, z_1)$ is not an irreducible component. Indeed, we have

$$Z_4^0 = \mathbf{V}(L_{222} + \langle f_{222}^{(4)} \rangle) = \mathbf{V}(L_{322})$$

by Remark 3.7(c) and Lemma 2.14(b), and hence

$$\operatorname{codim}_{(\mathbf{A}^3)_4} \mathbf{V}(L_{322}) = 3 + 2 + 2 = 7,$$

while by Lemma 2.17(b), the codimension of the irreducible component of S_4^0 is at most $4 + 2 = 6$. Thus, $S_4^0 \cap \mathbf{V}(y_1, z_1)$ is not an irreducible component of S_4^0 and, by Remark 3.7(a) and Proposition 3.9, the irreducible decomposition of S_4^0 is given by

$$Z_4^1 \cup Z_4^2 \cup Z_4^3.$$

□

Next lemma is one key point to prove that Z_m^0 is irreducible for $m \geq 5$.

Lemma 3.11. (a) *For $m = 5$, $Z_5^0 = \mathbf{V}(L_{322})$.*
(b) *For $m \geq 6$,*

$$Z_m^0 \cong \mathbb{A}^{11} \times S_{m-6}.$$

Proof. First note that, by Definition 3.6, Remark 2.6 and Remark 3.7(c), for $m \geq 5$, Z_m^0 is defined by

$$I_m^0 = L_{322} + \langle f^{(0)}, ..., f^{(m)} \rangle = L_{322} + \langle f_{322}^{(0)}, ..., f_{322}^{(m)} \rangle.$$

Moreover, by Lemma 2.14(a) and (b), we have

$$f_{322}^{(0)} = f_{322}^{(1)} = \cdots = f_{322}^{(5)} = 0.$$

(a) Clearly, we have $I_5^0 = L_{322} + \langle f_{322}^{(0)}, ..., f_{322}^{(5)} \rangle = L_{322}$. Hence $Z_5^0 = \mathbf{V}(L_{322})$.

(b) We have

$$I_m^0 = L_{322} + \langle f_{322}^{(6)}, ..., f_{322}^{(m)} \rangle$$

for $m \geq 6$. As in Notation 2.5, we write $\mathbf{x}_3 = x_3t^3 + x_4t^4 + \cdots + x_mt^m$, $\mathbf{y}_2 = y_2t^2 + y_3t^3 + \cdots + y_mt^m$ and $\mathbf{z}_2 = z_2t^2 + z_3t^3 + \cdots + z_mt^m$ and calculate $f(\mathbf{x}_3, \mathbf{y}_2, \mathbf{z}_2)$;

$$\begin{aligned}
f\left(\sum_{i=3}^m x_it^i, \sum_{i=2}^m y_it^i, \sum_{i=2}^m z_it^i\right) &= f\left(t^3 \sum_{i=3}^m x_it^{i-3}, t^2 \sum_{i=2}^m y_it^{i-2}, t^2 \sum_{i=2}^m z_it^{i-2}\right) \\
&= t^6 f\left(\sum_{i=3}^m x_it^{i-3}, \sum_{i=2}^m y_it^{i-2}, \sum_{i=2}^m z_it^{i-2}\right) \\
&= t^6 (f^{(0)}(x_3, y_2, z_2) + f^{(1)}(x_3, x_4, y_2, y_3, z_2, z_3)t + \cdots \\
&\quad + f^{(m-6)}(x_3, ..., x_{m-3}, y_2, ..., y_{m-4}, z_2, ..., z_{m-4})t^{m-6} + \cdots).
\end{aligned}$$

(Note that the second equality holds because f is weighted homogenous.) We set

$$f_6^{(l)} := f^{(l-6)}(x_3, ..., x_{l-3}, y_2, ..., y_{l-4}, z_2, ..., z_{l-4})$$

for $6 \leq l \leq m$, then $f_{322}^{(l)} = f_6^{(l)}$. Hence

$$\begin{aligned}
Z_m^0 &= \mathbf{V}(L_{322} + \langle f_{322}^{(6)}, ..., f_{322}^{(m)} \rangle) \\
&= \mathbf{V}(L_{322} + \langle f_6^{(6)}, ..., f_6^{(m)} \rangle) \cong \mathbb{A}^{11} \times S_{m-6}.
\end{aligned}$$

□

Now, we prove that Z_m^0 is irreducible of dimension $2m + 1$ for $m \geq 5$. To prove this claim, it suffices to show that S_m is irreducible of dimension $2(m + 1)$ for $m \geq 0$ by the previous lemma.

Remark 3.12. In [4, Corollary 10.2.9], the following statement is proven: For a locally complete intersection variety X and every positive integer m, the m-th jet scheme X_m of X is irreducible if and only if X has Mather-Jacobian canonical singularities.

We note that if X is a normal locally complete intersection variety, then the notion of Mather-Jacobian canonical singularities coincides with the notion of canonical singularities by Proposition 10.1.10 in [4]. Hence a singular surface of type D_4^0 is Mather-Jacobian canonical, and the above result can be applied.

For the reader's convenience, we prove the irreducibility of S_m by a direct calculation.

Proposition 3.13. (a) *For $m \geq 0$, S_m is irreducible of dimension $2(m+1)$ (or equivalently of codimension $m+1$ in $(\mathbb{A}^3)_m$).*

(b) *For $m \geq 5$, Z_m^0 is irreducible of dimension $2m+1$ (or equivalently of codimension $m+2$ in $(\mathbb{A}^3)_m$).*

Proof. By Remark 2.11,

$$S_m = \overline{(\pi_m^S)^{-1}(S_{\mathrm{sm}})} \cup S_m^0$$

and $\overline{(\pi_m^S)^{-1}(S_{\mathrm{sm}})}$ is irreducible of dimension $2(m+1)$, where $S_{\mathrm{sm}} = S - \mathrm{Sing}\, S$. By Lemma 2.17(a), in order to show the assertion (a) for S_m, it suffices to show that $\dim S_m^0 \leq 2m+1$.

For $0 \leq m \leq 4$, this holds by Proposition 3.10. Thus (a) holds for $0 \leq m \leq 4$.

Before proving the statements for $m \geq 5$, we note that, by Proposition 3.9, we have

$$S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3,$$

and, by Remark 3.7, Z_m^1, Z_m^2 and Z_m^3 are irreducible of dimension $2m+1$. Thus it suffices to show that the remaining part of S_m^0, i.e. Z_m^0, is irreducible of dimension $2m+1$ for $m \geq 5$, i.e. the statement (b).

For $m = 5$, we have $Z_5^0 = \mathbf{V}(L_{322})$ and Z_5^0 is irreducible of dimension $3(5+1) - 7 = 11 (= 2 \times 5 + 1)$ by Lemma 3.11(a). Thus S_5 is irreducible of dimension 12.

For $m \geq 6$, we assume that (a) holds for $S_{m'}$ with $0 \leq m' < m$ and show the assertion for S_m and Z_m^0. By Lemma 3.11(b), we have $Z_m^0 \cong \mathbb{A}^{11} \times S_{m-6}$. By the inductive hypothesis, S_{m-6} is irreducible of dimension $2(m-6+1) = 2m-10$. Hence Z_m^0 is irreducible of dimension

$$\dim Z_m^0 = \dim \mathbb{A}^{11} \times S_{m-6} = 11 + (2m-10) = 2m+1.$$

Thus S_m is irreducible of dimension $2(m+1)$. □

Now, we give the irreducible decomposition of S_m^0 for $m \geq 5$. In the following theorem, note that the number of irreducible components of the singular fiber is constant.

Theorem 3.14. *For $m \geq 5$, the irreducible decomposition of the singular fiber S_m^0 is given by*

$$S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3.$$

Proof. First of all, we note that, for $0 \leq i \leq 3$, the closed subsets Z_m^i are irreducible of dimension $2m+1$ by Remark 3.7 and Proposition 3.13(b). Moreover, by Proposition 3.9, we have

$$S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3$$

for $m \geq 5$. Hence all that remains is to check that Z_m^i are pairwise distinct for $0 \leq i \leq 3$. For $m \geq 5$, we have $Z_m^0 \cap (\mathbf{D}(y_1) \cup \mathbf{D}(z_1)) = \emptyset$ while $Z_m^i \cap (\mathbf{D}(y_1) \cup \mathbf{D}(z_1)) \neq \emptyset$ for $1 \leq i \leq 3$, so Z_m^0 is different from Z_m^1, Z_m^2 and Z_m^3. Moreover, by Proposition 3.9, Z_m^1, Z_m^2 and Z_m^3 are pairwise distinct. This completes the proof. □

In the following, we determine the inclusion relation between the intersections of two irreducible components of the singular fiber of a singular surface of type D_4^0.

Theorem 3.15. *Let k be an algebraically closed field of characteristic 2, $S \subset \mathbb{A}^3$ the surface defined by $f = x^2 + y^2 z + yz^2$ in the affine space over k, S_m^0 the singular fiber of the m-th jet scheme S_m with $m \geq 5$ and $Z_m^0, \ldots, Z_m^3$ its irreducible components as in Definition 3.6.*

(a) *For $0 \leq i < j \leq 3$, $Z_m^i \cap Z_m^j \subsetneq Z_m^0$.*

(b) *For* $1 \leq i, j \leq 3$ *with* $i \neq j$, $Z_m^0 \cap Z_m^i \not\subseteq Z_m^0 \cap Z_m^j$.
(c) *For* $1 \leq i, j \leq 3$ *with* $i \neq j$, $Z_m^i \cap Z_m^j \subsetneq Z_m^0 \cap Z_m^i$.
(d) *For* $1 \leq i < j \leq 3$ *and* $1 \leq l \leq 3$, $Z_m^0 \cap Z_m^l \not\subseteq Z_m^i \cap Z_m^j$.

In particular, for $0 \leq i < j \leq 3$, $Z_m^i \cap Z_m^j$ *is maximal in* $\{Z_m^i \cap Z_m^j \mid i, j \in \{0,1,2,3\}, i \neq j\}$ *with respect to the inclusion relation if and only if* $(i, j) = (0,1), (0,2), (0,3)$.

Remark 3.16. Over algebraically closed fields of characteristic 0, the same statment was proved in [5, Theorem 3.17].

Proof. (a) By looking at the dimensions, we have $Z_m^0 \cap Z_m^j \subsetneq Z_m^0$ for $j = 1, 2, 3$. Hence the assertion holds if $i = 0$, so we may assume that $(i, j) = (1, 2)$ by Lemma 3.8. From the definitions of Z_m^i (Definition 3.6), it suffices to check that $\sqrt{I_m^1 + I_m^2} \supseteq I_m^0$. By the definitions of the ideals J_m^1 and J_m^2, we have $y_1 \in J_m^1$ and $z_1 \in J_m^2$, hence $\sqrt{J_m^1 + J_m^2} \supseteq L_{222}$. Therefore $\sqrt{I_m^1 + I_m^2} \supseteq \sqrt{J_m^1 + J_m^2} \supseteq I_m^0$. By looking at the dimensions, we have $Z_m^1 \cap Z_m^2 \subsetneq Z_m^0$.

(b) By Lemma 3.8, we only have to show that $Z_m^0 \cap Z_m^1 \not\subseteq Z_m^0 \cap Z_m^2$. Let us consider the jet $\gamma = (0, 0, t^2) \in S_m$. We prove the following two claims:

(i) $\gamma \in Z_m^0 \cap Z_m^1$,
(ii) $\gamma \notin Z_m^0 \cap Z_m^2$.

(i) We can easily check that $\gamma \in Z_m^0$ by Definition 3.6. Let us consider the family

$$\gamma_s := (0, 0, st + t^2) \text{ for } s \in k.$$

If $s \neq 0$, then we have $\gamma_s \in \mathbf{V}(J_m^1)$ and $\gamma_s \in \mathbf{D}(z_1)$. Thus, taking the Zariski closure of $\mathbf{V}(J_m^1) \cap \mathbf{D}(z_1)$, we have $\gamma_0 \in \overline{\mathbf{V}(J_m^1) \cap \mathbf{D}(z_1)} = Z_m^1$.

(ii) We show that $z_2 \in \sqrt{I_m^0 + I_m^2}$, and then $\gamma \notin Z_m^0 \cap Z_m^2$. By Lemma 2.14,

$$f_{212}^{(5)} = \sum_{u \geq 2, 2u=5} x_u^2 + \sum_{v \geq 1, w \geq 2, 2v+w=5} y_v^2 z_w + \sum_{v \geq 1, w \geq 2, v+2w=5} y_v z_w^2 = y_1^2 z_3 + y_1 z_2^2 = y_1(y_1 z_3 + z_2^2).$$

Then we have $y_1 z_3 + z_2^2 \in J_m^2 \cdot (R_m)_{y_1} \cap R_m = I_m^2$, so

$$z_2^2 = -y_1 z_3 + (y_1 z_3 + z_2^2) \in I_m^0 + I_m^2.$$

Hence we have $z_2 \in \sqrt{I_m^0 + I_m^2}$ and $\gamma \notin Z_m^0 \cap Z_m^2$. Therefore, we have $Z_m^0 \cap Z_m^i \not\subseteq Z_m^0 \cap Z_m^j$, or equivalently $Z_m^0 \cap Z_m^i \not\subseteq Z_m^j$, for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

(c) By the assertion (a), we have $Z_m^i \cap Z_m^j \subseteq Z_m^0 \cap Z_m^i$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. If $Z_m^i \cap Z_m^j = Z_m^0 \cap Z_m^i$, then

$$Z_m^0 \cap Z_m^i = Z_m^0 \cap Z_m^i \cap Z_m^j \subseteq Z_m^0 \cap Z_m^j,$$

a contradiction to (b). Thus, $Z_m^i \cap Z_m^j \subsetneq Z_m^0 \cap Z_m^i$

(d) Take any $i, j, l \in \{1, 2, 3\}$ with $i \neq j$ and $j \neq l$ (not necessarily $i \neq l$). If $Z_m^0 \cap Z_m^l \subseteq Z_m^i \cap Z_m^j$, then

$$Z_m^0 \cap Z_m^l \subseteq Z_m^0 \cap Z_m^i \cap Z_m^j \subseteq Z_m^0 \cap Z_m^j,$$

a contradiction to (b). Hence $Z_m^0 \cap Z_m^l \not\subseteq Z_m^i \cap Z_m^j$. □

Now we define a graph Γ from the information of S_m^0 as follows.

Construction 3.17 ([5, Construction 2.15])**.** The graph $\Gamma(S_m^0) = (V, E)$ is constructed as follows.

- V is the set of irreducible components of S_m^0.

- E is the set of all maximal elements of $\{Z_m^i \cap Z_m^j \mid i, j \in \{0,1,2,3\}, i \neq j\}$, and $Z_m^i \cap Z_m^j \in E$ connects Z_m^i and Z_m^j.

In other words, there is given an edge between Z_m^i and Z_m^j if and only if $Z_m^i \cap Z_m^j$ is maximal among the intersections of two distinct irreducible components.

Corollary 3.18. *For a singular surface S of type D_4^0 and $m \geq 5$, the graph $\Gamma(S_m^0)$ obtained by Construction 3.17 is isomorphic to the resolution graph of S.*

Proof. By Theorem 3.14 and Theorem 3.15, $\Gamma(S_m^0)$ is the pair of $V = \{Z_m^0, Z_m^1, Z_m^2, Z_m^3\}$ and $E = \{Z_m^0 \cap Z_m^1, Z_m^0 \cap Z_m^2, Z_m^0 \cap Z_m^3\}$. Hence $\Gamma(S_m^0)$ can be described as

$$\begin{tikzcd}
 & & Z_m^2 \\
Z_m^1 \arrow[r, dash] & Z_m^0 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & Z_m^3,
\end{tikzcd}$$

which is a Dynkin diagram of type D_4. □

4. Jet schemes of a singular surface of type D_4^1

In this section, we prove the statements of Theorem 1.1 and Theorem 1.2 on the singular surface of type D_4^1. The proof goes mostly in the same way as in [7] and Theorem 3.14, but we need more elaborate analysis to show that Z_m^0 is irreducible.

Let $g = x^2 + y^2 z + yz^2 + xyz$ and $S = \mathbf{V}(g) \subseteq \mathbb{A}^3$, which has a singularity of type D_4^1 at 0 ([1, Section 3]). Furthermore, let $R_m = k[x_0, ..., x_m, y_0, ..., y_m, z_0, ..., z_m]$, S_m the m-th jet scheme of S and S_m^0 the singular fiber of S_m. Here, we note that the equation $g = x^2 + y^2 z + yz^2 + xyz$ is not weighted homogenous, and hence a certain part of the arguments of Mourtada [7] does not work in this case. More specifically, we used the fact that the equation is weighted homogenous to prove Lemma 3.11(b), but in this case we cannot use this idea. In this section, we focus on certain subsets of Z_m^0 and their dimensions (or equivalently codimensions) to prove the irreducibility of Z_m^0.

Let the ideal $L_{pqr} := \langle x_0, ..., x_{p-1}, y_0, ..., y_{q-1}, z_0, ..., z_{r-1} \rangle \subseteq R_m = k[x_0, ..., x_m, y_0, ..., y_m, z_0, ..., z_m]$ for $0 < p, q, r \leq m + 1$ be as in the previous section.

Definition 4.1. For $m \geq 1$, we consider the following ideals in R_m:

$$\begin{aligned}
J_m^1 &= L_{211} + \langle y_1 \rangle &&+ \langle g^{(0)}, ..., g^{(m)} \rangle = L_{221} + \langle g^{(0)}, ..., g^{(m)} \rangle, \\
J_m^2 &= L_{211} + \langle z_1 \rangle &&+ \langle g^{(0)}, ..., g^{(m)} \rangle = L_{212} + \langle g^{(0)}, ..., g^{(m)} \rangle, \\
J_m^3 &= L_{211} + \langle y_1 + z_1 \rangle &&+ \langle g^{(0)}, ..., g^{(m)} \rangle.
\end{aligned}$$

By Remark 2.9 and $J_m^i \supset L_{111}$ for $i = 1, 2, 3$, these ideals include the defining ideal of the singular fiber, and hence correspond to closed subsets in the singular fiber S_m^0.

We have the following symmetries:

Notation 4.2. Let φ_1 and φ_2 be the automorphisms of R_m defined by

$$\varphi_1 : \begin{cases} x_i \mapsto x_i, \\ y_i \mapsto z_i, \\ z_i \mapsto y_i, \end{cases} \qquad \varphi_2 : \begin{cases} x_i \mapsto x_i, \\ y_i \mapsto y_i, \\ z_i \mapsto x_i + y_i + z_i. \end{cases}$$

These automorphisms φ_1 and φ_2 induce ring isomorphisms

$$(\varphi_1)_{z_1} : (R_m)_{z_1} \to (R_m)_{y_1} \text{ and } (\varphi_2)_{y_1} : (R_m)_{y_1} \to (R_m)_{y_1}.$$

We write the isomorphisms corresponding to φ_1, φ_2, $(\varphi_1)_{z_1}$ and $(\varphi_2)_{y_1}$ as ψ_1, ψ_2, $(\psi_1)_{z_1}$ and $(\psi_2)_{y_1}$, respectively. For simplicity, we write $(\varphi_1)_{z_1}$, $(\varphi_2)_{y_1}$, $(\psi_1)_{z_1}$ and $(\psi_2)_{y_1}$ as φ_1, φ_2, ψ_1 and ψ_2.

Lemma 4.3. (a) *For $m \geq 1$, $i \in \{0, ..., m\}$ and $k = 1, 2$, φ_k preserve $g^{(i)}$ i.e.,*

$$\varphi_k(g^{(i)}) = g^{(i)}$$

in R_m. In particular, the morphisms ψ_k preserve S_m.

(b) *For $m \geq 1$,*

$$\varphi_1(J^1_m \cdot (R_m)_{z_1}) = J^2_m \cdot (R_m)_{y_1}$$

and

$$\varphi_2(J^2_m \cdot (R_m)_{y_1}) = J^3_m \cdot (R_m)_{y_1}.$$

(c) *The morphisms φ_1, φ_2, ψ_1 and ψ_2 preserve the union, the intersection and the inclusion relations of sets.*

Proof. The proofs of the assertions (b) and (c) are the same as those of Lemma 3.4(b) and (c), so we only check the assertion (a).

Note that the automorphisms φ_1 and φ_2 are induced by the automorphisms of $k[x, y, z]$ defined by

$$\overline{\varphi_1} = \begin{cases} x \mapsto x, \\ y \mapsto z, \\ z \mapsto y, \end{cases} \quad \overline{\varphi_2} = \begin{cases} x \mapsto x, \\ y \mapsto y, \\ z \mapsto x + y + z, \end{cases}$$

and $\overline{\varphi_1}(g) = x^2 + z^2y + zy^2 + xzy = g$ and

$$\begin{aligned} \overline{\varphi_2}(g) &= x^2 + y^2(x + y + z) + y(x + y + z)^2 + xy(x + y + z) \\ &= x^2 + y(x + y + z)(y + (x + y + z) + x) \\ &= x^2 + y(x + y + z)z \qquad\qquad = g, \end{aligned}$$

where $y + (x + y + z) + x = 2x + 2y + z = z$ since k is a field of characteristic 2. In the same way as in Lemma 3.4(a), φ_k preserves the polynomials $g^{(i)}$ $(i \in \{0, ..., m\})$, and the induced automorphism ψ_k preserves S_m. □

Now, we prove the following lemma. This lemma is one key point to prove that Z^i_m's (these will be defined in Definition 4.7) are irreducible for $0 \leq i \leq 3$.

Lemma 4.4. *For $m, p, q, r \in \mathbb{Z}_{>0}$ with $m \geq 2p$, the following hold.*

(a) *If $p \geq q > r$ and $2p = q + 2r$, then $\overline{(S_m \cap \mathbf{V}(L_{pqr})) \cap \mathbf{D}(z_r)}$ is irreducible and has codimension $m - p + q + r + 1 = m + p - r + 1$ in $(\mathbb{A}^3)_m$.*

(b) *If $p \geq r > q$ and $2p = 2q + r$, then $\overline{(S_m \cap \mathbf{V}(L_{pqr})) \cap \mathbf{D}(y_q)}$ is irreducible and has codimension $m - p + q + r + 1 = m + p - q + 1$ in $(\mathbb{A}^3)_m$.*

(c) *If $p > q = r$ and $2p > 3q$, then*

$$\overline{(S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)) \cap \mathbf{D}(y_q)} = \overline{(S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)) \cap \mathbf{D}(z_q)}$$

holds, and this set is irreducible of codimension $m + p - q + 1$ in $(\mathbb{A}^3)_m$.

(d) *If $p > q = r$ and $2p = 3q$, then*

$$\overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q)} = \overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(z_q)}$$

holds, and this set is irreducible of codimension $m - p + 2q + 1 = m + p - q + 1$ in $(\mathbb{A}^3)_m$. If furthermore $m = 2p$, then

$$\overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q)} = S_m \cap \mathbf{V}(L_{pqq}) = \mathbf{V}(L_{pqq} + \langle g^{(2p)}_{pqq} \rangle).$$

Proof. Before proving the lemma, note that

$$S_m \cap \mathbf{V}(L_{pqr}) = \mathbf{V}(L_{pqr} + \langle g^{(0)}, ..., g^{(m)} \rangle) = \mathbf{V}(L_{pqr} + \langle g^{(0)}_{pqr}, ..., g^{(m)}_{pqr} \rangle)$$

by Remark 2.6.

(a) By Lemma 2.14(e), for an integer l satisfying $2p \leq l \leq m$, there exists a polynomial $h^{(l)} \in k[x_p, ..., x_l, y_q, ..., y_{l-2r-1}, z_r, ..., z_l]$ such that

$$g^{(l)}_{pqr} = y_{l-2r} z_r^2 + h^{(l)}.$$

On the other hand, if $l < 2p$, then we have $l < 2p = q + 2r < 2q + r$. Hence by Lemma 2.14(a), $g^{(l)}_{pqr} = 0$ and $(S_m \cap \mathbf{V}(L_{pqr})) \cap \mathbf{D}(z_r)$ is defined by

$$(1)\ (L_{pqr} + \langle g^{(0)}, ..., g^{(m)} \rangle) \cdot (R_m)_{z_r} = L_{pqr} \cdot (R_m)_{z_r} + \left\langle y_{2p-2r} + \frac{h^{(2p)}}{z_r^2}, y_{2p+1-2r} + \frac{h^{(2p+1)}}{z_r^2}, ..., y_{m-2r} + \frac{h^{(m)}}{z_r^2} \right\rangle$$

in $\mathbf{D}(z_r) \subset (\mathbb{A}^3)_m$. This ideal is prime of height $p + q + r + (m - 2p + 1) = m - p + q + r + 1 = m + p - r + 1$. Therefore, $\overline{(S_m \cap \mathbf{V}(L_{pqr})) \cap \mathbf{D}(z_r)}$ is irreducible of codimension $m + p - r + 1$.

(b) Using the automorphism φ_1, we see that (b) follows from (a).

(c) First, we note that $y_q = z_q$ holds on $S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)$, so

$$\gamma \in (S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)) \cap \mathbf{D}(y_q) \Leftrightarrow \gamma \in (S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)) \cap \mathbf{D}(z_q).$$

Hence we have

$$\overline{(S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)) \cap \mathbf{D}(y_q)} = \overline{(S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)) \cap \mathbf{D}(z_q)}.$$

Next, we prove that the closed subset $\overline{(S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)) \cap \mathbf{D}(y_q)}$ is irreducible of codimension $m+p-q+1$. It is enough to show that $(L_{pqq}+\langle y_q+z_q \rangle+\langle g^{(0)}, ..., g^{(m)} \rangle)\cdot(R_m)_{y_q}$ is prime of height $m+p-q+1$. Let us look at $g^{(l)}_{pqq}$ for l with $l \leq m$. If $l < 3q$, then $l < 3q < 2p$. So $g^{(l)}_{pqq} = 0$ by Lemma 2.14(a). If $l = 3q$, then $g^{(3q)}_{pqq} = y_q^2 z_q + y_q z_q^2$ by Lemma 2.14(d), and $g^{(3q)}_{pqq} \in \langle y_q + z_q \rangle$. Finally, if $3q < l \leq m$, $T_z(g^{(l)}_{pqq}) = y_q^2 z_{l-2q}$ by Lemma 2.14(h). So there exists a polynomial $h^{(l)} \in k[x_p, ..., x_m, y_q, ..., y_m, z_q, ..., z_{l-2q-1}]$ such that

$$g^{(l)}_{pqq} = y_q^2 z_{l-2q} + h^{(l)}.$$

Therefore, $(S_m \cap \mathbf{V}(L_{pqq} + \langle y_q + z_q \rangle)) \cap \mathbf{D}(y_q)$ is defined by

$$\begin{aligned}&(L_{pqq} + \langle y_q + z_q \rangle + \langle g^{(0)}, ..., g^{(m)} \rangle) \cdot (R_m)_{y_q} \\ =&(L_{pqq} + \langle y_q + z_q \rangle) \cdot (R_m)_{y_q} + \left\langle z_{q+1} + \frac{h^{(3q+1)}}{y_q^2}, z_{q+2} + \frac{h^{(3q+2)}}{y_q^2}, ..., z_{m-2q} + \frac{h^{(m)}}{y_q^2} \right\rangle\end{aligned}$$

and this ideal is prime of height $p + q + q + 1 + (m - 3q) = m + p - q + 1$.

(d) First, we prove the following claim.

Claim 4.5. *$\overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q)}$ is irreducible of codimension $m + p - q + 1$.*

We think of the defining ideal $L_{pqq} + \langle g^{(0)}, ..., g^{(m)} \rangle$ of $S_m \cap \mathbf{V}(L_{pqq})$ as

$$L_{pqq} + \langle g^{(0)}, ..., g^{(2p)} \rangle + \langle g^{(2p+1)}, ..., g^{(m)} \rangle.$$

By Lemma 2.14(a) and Lemma 2.14(c),

$$L_{pqq} + \langle g^{(0)}, ..., g^{(2p)} \rangle = L_{pqq} + \langle g^{(2p)}_{pqq} \rangle,$$

with $g^{(2p)}_{pqq} = x_p^2 + y_q^2 z_q + y_q z_q^2$ and by Notation 2.5,

$$\{g^{(2p+1)}_{pqq}, ..., g^{(m)}_{pqq}\} \subset k[x_p, ..., x_m, y_q, ..., y_m, z_q, ..., z_m].$$

Now, we consider the following residue ring:

$$\begin{aligned} Q &= k[x_0, ..., x_p, y_0, ..., y_q, z_0, ..., z_q]/\langle x_0, ..., x_{p-1}, y_0, ..., y_{q-1}, z_0, ..., z_{q-1}, g^{(2p)}_{pqq}\rangle \\ &\cong k[x_p, y_q, z_q]/\langle g^{(2p)}_{pqq}\rangle. \end{aligned}$$

Since $g^{(2p)}_{pqq}$ is irreducible in $k[x_p, y_q, z_q]$ by Remark 2.15, this ring is an integral domain. Since

$$\begin{aligned} Q_m &:= Q[x_{p+1}, ..., x_m, y_{q+1}, ..., y_m, z_{q+1}, ..., z_m] \\ &\cong R_m/(\langle x_0, ..., x_{p-1}, y_0, ..., y_{q-1}, z_0, ..., z_{q-1}, g^{(2p)}_{pqq}\rangle), \end{aligned}$$

$L_{pqq} + \langle g^{(2p)}_{pqq}\rangle$ is prime of height $p + q + q + 1 = 3p - q + 1$ and the claim holds if $m = 2p$. Moreover, we have $y_q \notin \langle g^{(2p)}_{pqq}\rangle$ and $L_{pqq} + \langle g^{(2p)}_{pqq}\rangle$ is prime, so $\overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q)} = S_m \cap \mathbf{V}(L_{pqq})$.

In the following, we assume $m > 2p$. Let $\overline{g^{(2p+1)}_{pqq}}, ..., \overline{g^{(m)}_{pqq}}$ and $\overline{y_q}$ be the images of $g^{(2p+1)}_{pqq}, ..., g^{(m)}_{pqq}$ and y_q by the natural surjection $R_m \to Q_m$. What we have to show is that $\langle \overline{g^{(2p+1)}_{pqq}}, ..., \overline{g^{(m)}_{pqq}}\rangle \cdot (Q_m)_{\overline{y_q}}$ is prime of height $m - 2p$. Here, we note that $\overline{y_q} \neq 0$ in Q_m since $y_q \notin \langle g^{(2p)}_{pqq}\rangle$ and that $Q_{\overline{y_q}}$ is also an integral domain. Assume $2p + 1 \leq l \leq m$. Then we have $l \geq 2p + 1 > 2p = 3q$. Hence there exists $h^{(l)} \in Q[x_{p+1}, ..., x_l, y_{q+1}, ..., y_l, z_{q+1}, ..., z_{l-2q-1}]$ such that $\overline{g^{(l)}_{pqq}} = \overline{y_q}^2 z_{l-2q} + h^{(l)}$ by Lemma 2.14(h). Therefore, from

$$\langle \overline{g^{(2p+1)}_{pqq}}, ..., \overline{g^{(m)}_{pqq}}\rangle \cdot (Q_m)_{\overline{y_q}} = \left\langle z_{2p-2q+1} + \frac{h^{(2p+1)}}{\overline{y_q}^2}, z_{2p-2q+2} + \frac{h^{(2p+2)}}{\overline{y_q}^2}, ..., z_{m-2q} + \frac{h^{(m)}}{\overline{y_q}^2} \right\rangle,$$

it follows that $\langle \overline{g^{(2p+1)}_{pqq}}, ..., \overline{g^{(m)}_{pqq}}\rangle \cdot (Q_m)_{\overline{y_q}}$ is prime of height $m - 2p$ in

$$(Q_m)_{\overline{y_q}} \cong Q_{\overline{y_q}}[x_{p+1}, ..., x_m, y_{q+1}, ..., y_m, z_{q+1}, ..., z_m].$$

This proves the claim.

Finally, we prove

$$\overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q)} = \overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(z_q)}.$$

Note that the right-hand side is also irreducible by the symmetry. From $g(0, t^q, t^q) = 0$, it follows that $(0, t^q, t^q) \in (S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q z_q)$. Since an irreducible closed set is the closure of its nonempty open subset, we have

$$\overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q)} = \overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q z_q)} = \overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(z_q)}.$$

□

Corollary 4.6. *For $m \geq 3$, the ideals $J^1_m \cdot (R_m)_{z_1}$, $J^2_m \cdot (R_m)_{y_1}$ and $J^3_m \cdot (R_m)_{y_1}$ are prime, and the closed subsets $\overline{\mathbf{V}(J^1_m) \cap \mathbf{D}(z_1)}$, $\overline{\mathbf{V}(J^2_m) \cap \mathbf{D}(y_1)}$ and $\overline{\mathbf{V}(J^3_m) \cap \mathbf{D}(y_1)}$ are irreducible of dimension $2m + 1$ (or equivalently of codimension $m + 2$ in $(\mathbb{A}^3)_m$).*

Proof. For $m = 3$, it is easy to see that J^1_3, J^2_3 and J^3_3 are prime of height 5. For instance, by Lemma 2.14(a), $f^{(0)}_{221} = f^{(1)}_{221} = f^{(2)}_{221} = f^{(3)}_{221} = 0$. Hence $J^1_3 = L_{221}$ is prime of height 5. Similarly for J^2_3 and J^3_3.

For $m \geq 4$, we apply Lemma 4.4(a) with $p = q = 2$ and $r = 1$ to show that $\overline{\mathbf{V}(J^1_m) \cap \mathbf{D}(z_1)}$ is irreducible of codimension

$$m + 2 - 1 + 1 = m + 2.$$

We see that $J^1_m \cdot (R_m)_{z_1}$ is prime from (1) in the proof of Lemma 4.4(a).

Using Lemma 4.3(b), we also see that $J^2_m \cdot (R_m)_{y_1}$ and $J^3_m \cdot (R_m)_{y_1}$ are prime and $\overline{\mathbf{V}(J^2_m) \cap \mathbf{D}(y_1)}$ and $\overline{\mathbf{V}(J^3_m) \cap \mathbf{D}(y_1)}$ are irreducible of codimension $m + 2$. □

Now we define ideals in R_m that will turn out to be defining ideals of the irreducible components of S^0_m for $m \geq 5$.

Definition 4.7. For $m \geq 1$, we define

$$
\begin{aligned}
I_m^0 &= L_{222} + \langle g^{(0)}, ..., g^{(m)} \rangle, \\
I_m^1 &= J_m^1 \cdot (R_m)_{z_1} \cap R_m, \\
I_m^2 &= J_m^2 \cdot (R_m)_{y_1} \cap R_m, \\
I_m^3 &= J_m^3 \cdot (R_m)_{y_1} \cap R_m.
\end{aligned}
$$

Furthermore, we define closed subsets

$$Z_m^i = \mathbf{V}(I_m^i)$$

for $0 \leq i \leq 3$.

Remark 4.8. By Lemma 2.14(b), we have $f_{222}^{(4)} = x_2^2$. Hence, if $m \geq 4$,

$$Z_m^0 = \mathbf{V}(L_{322} + \langle g^{(0)}, ..., g^{(m)} \rangle) = S_m \cap \mathbf{V}(L_{322}).$$

Assume $2p = 3q$ and $m \geq 2p$. By Remark 2.6 and Lemma 2.14(a),

$$L_{pqq} + \langle g^{(0)}, ..., g^{(m)} \rangle = L_{pqq} + \langle g_{pqq}^{(0)}, ..., g_{pqq}^{(m)} \rangle = L_{pqq} + \langle g_{pqq}^{(2p)}, ..., g_{pqq}^{(m)} \rangle.$$

The same arguments show that

$$L_{pqq} + \langle g^{(0)}, ..., g^{(2p-1)} \rangle = L_{pqq}.$$

Lemma 4.9. *For $m \geq 5$, Z_m^0 is irreducible of dimension $2m + 1$ (or equivalently of codimension $m + 2$ in $(\mathbb{A}^3)_m$).*

The strategy of the proof of this lemma is as follows: Let $u = \lfloor m/6 \rfloor$. We decompose Z_m^0 into $Z_m^0 \cap \left(\mathbf{V}(L_{3,u',u'}) \cap (\mathbf{D}(y_{u'}) \cup \mathbf{D}(z_{u'}))\right)$ for $u' < 2u$ and $Z_m^0 \cap \mathbf{V}(L_{3,2u,2u})$ and calculate their codimensions. The codimensions of the former are calculated in the same way as in Lemma 4.4. As for the latter, we calculate these sets in Proposition 4.10. On the other hand, we have an upper bound for the codimension of the irreducible components of Z_m^0. Then we can argue as in [6] to conclude that Z_m^0 is irreducible.

Before proving Lemma 4.9, we show the following proposition.

Proposition 4.10. *Let p, q and u be positive integers with $p = 3u$ and $q = 2u$, and assume $2p \leq m < 2(p + 3)$. We set*

$$
\begin{aligned}
U_m &:= \mathbf{V}(L_{pqq} + \langle g_{pqq}^{(2p)}, ..., g_{pqq}^{(m)} \rangle) &&= S_m \cap \mathbf{V}(L_{pqq}), \\
V_m &:= \mathbf{V}(L_{p,q+1,q+1} + \langle g_{p,q+1,q+1}^{(2p)}, ..., g_{p,q+1,q+1}^{(m)} \rangle) &&= S_m \cap \mathbf{V}(L_{p,q+1,q+1}), \\
W_m &:= \mathbf{V}(L_{p+2,q+2,q+2} + \langle g_{p+2,q+2,q+2}^{(2p)}, ..., g_{p+2,q+2,q+2}^{(m)} \rangle) &&= S_m \cap \mathbf{V}(L_{p+2,q+2,q+2}).
\end{aligned}
$$

(a) *The codimension of the closed subset W_m in $(\mathbb{A}^3)_m$ is greater than $m + u + 1$.*
(b) *The codimension of the closed subset V_m in $(\mathbb{A}^3)_m$ is greater than $m + u + 1$.*
(c) *U_m is irreducible of codimension $m + u + 1$ in $(\mathbb{A}^3)_m$, and*

$$U_m = \overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(y_q)} = \overline{(S_m \cap \mathbf{V}(L_{pqq})) \cap \mathbf{D}(z_q)}.$$

Proof. We fix u (and p, q), and deal with the cases $m = 2p, 2p + 1, ..., 2p + 5$ in order.

(a) The cases $m \leq 2p + 3$. By Lemma 2.14(a), we have

$$g_{p+2,q+2,q+2}^{(2p)} = \cdots = g_{p+2,q+2,q+2}^{(2p+3)} = 0.$$

Hence

$$L_{p+2,q+2,q+2} + \langle g_{p+2,q+2,q+2}^{(2p)}, ..., g_{p+2,q+2,q+2}^{(m)} \rangle = L_{p+2,q+2,q+2}$$

and

$$\operatorname{codim}_{(\mathbb{A}^3)_m} W_m = p + 2 + q + 2 + q + 2 = 7u + 6 > 7u + 4 \geq m + u + 1.$$

The case $m = 2p + 4$. By Lemma 2.14(b), $g^{(2p+4)}_{p+2,q+2,q+2} = x^2_{p+2}$. Hence

$$W_{2p+4} = \mathbf{V}(L_{p+2,q+2,q+2} + \langle x^2_{p+2}\rangle) = \mathbf{V}(L_{p+3,q+2,q+2}).$$

Thus, W_{2p+4} is irreducible of codimension

$$p + 3 + q + 2 + q + 2 = 7u + 7 > 7u + 5 = (2p + 4) + u + 1$$

in $(\mathbb{A}^3)_m$.

The case $m = 2p + 5$. We note that

$$W_{2p+5} = (\pi^S_{2p+5,2p+4})^{-1}(W_{2p+4}) = \mathbf{V}(L_{p+3,q+2,q+2} + \langle g^{(2p+5)}_{p+3,q+2,q+2}\rangle).$$

By Lemma 2.14(a), $g^{(2p+5)}_{p+3,q+2,q+2} = 0$. Hence W_{2p+5} is irreducible of codimension

$$p + 3 + q + 2 + q + 2 = 7u + 7 > 7u + 6 = (2p + 5) + u + 1$$

in $(\mathbb{A}^3)_m$. This completes the proof of (a).

(b) The case $m = 2p$. By Lemma 2.14(b), $g^{(2p)}_{p,q+1,q+1} = x^2_p$. Hence

$$V_{2p} = \mathbf{V}(L_{p,q+1,q+1} + \langle x^2_p\rangle) = \mathbf{V}(L_{p+1,q+1,q+1})$$

and V_{2p} is of codimension $p + 1 + q + 1 + q + 1 = 7u + 3 > 7u + 1 = 2p + u + 1$.

We note that, for $m \geq 2p + 1$,

$$V_m = (\pi^S_{m,2p})^{-1}(V_{2p}) = \mathbf{V}(L_{p+1,q+1,q+1} + \langle g^{(2p+1)}_{p+1,q+1,q+1}, \dots, g^{(m)}_{p+1,q+1,q+1}\rangle).$$

The case $m = 2p + 1$. By Lemma 2.14(a), $g^{(2p+1)}_{p+1,q+1,q+1} = 0$ and V_{2p+1} is of codimension

$$p + 1 + q + 1 + q + 1 = 7u + 3 > 7u + 2 = (2p + 1) + u + 1.$$

The case $m = 2p + 2$. By Lemma 2.14(b), $g^{(2p+2)}_{p+1,q+1,q+1} = x^2_{p+1}$. Hence

$$V_{2p+2} = \mathbf{V}(L_{p+1,q+1,q+1} + \langle x^2_{p+1}\rangle) = \mathbf{V}(L_{p+2,q+1,q+1})$$

and V_{2p+2} is of codimension $p + 2 + q + 1 + q + 1 = 7u + 4 > 7u + 3 = (2p + 2) + u + 1$.

The case $m = 2p + 3$. By Lemma 2.14(d), $g^{(2p+3)}_{p+2,q+1,q+1} = y_{q+1}z_{q+1}(y_{q+1} + z_{q+1})$. Hence

$$\begin{aligned} V_{2p+3} &= \mathbf{V}(L_{p+2,q+1,q+1} + \langle y_{q+1}z_{q+1}(y_{q+1} + z_{q+1})\rangle) \\ &= \mathbf{V}(L_{p+2,q+2,q+1}) \cup \mathbf{V}(L_{p+2,q+1,q+2}) \cup \mathbf{V}(L_{p+2,q+1,q+1} + \langle y_{q+1} + z_{q+1}\rangle). \end{aligned}$$

Thus

$$\operatorname{codim}_{(\mathbb{A}^3)_m} V_{2p+3} = 7u + 5 > 7u + 4 = (2p + 3) + u + 1.$$

The cases $m \geq 2p + 4$. Since

$$V_m = (\pi^S_{m,2p+3})^{-1}(V_{2p+3}) =$$
$$(S_m \cap \mathbf{V}(L_{p+2,q+2,q+1})) \cup (S_m \cap \mathbf{V}(L_{p+2,q+1,q+2})) \cup (S_m \cap \mathbf{V}(L_{p+2,q+1,q+1} + \langle y_{q+1} + z_{q+1}\rangle))$$

and $(\mathbb{A}^3)_m = \mathbf{D}(y_{q+1}) \cup \mathbf{D}(z_{q+1}) \cup \mathbf{V}(y_{q+1}, z_{q+1})$, we have

$$\begin{aligned} V_m = {} & \overline{(S_m \cap \mathbf{V}(L_{p+2,q+2,q+1})) \cap \mathbf{D}(z_{q+1})} \cup \overline{(S_m \cap \mathbf{V}(L_{p+2,q+1,q+2})) \cap \mathbf{D}(y_{q+1})} \cup \\ & \overline{(S_m \cap \mathbf{V}(L_{p+2,q+1,q+1} + \langle y_{q+1} + z_{q+1}\rangle)) \cap \mathbf{D}(y_{q+1})} \cup \\ & \overline{(S_m \cap \mathbf{V}(L_{p+2,q+1,q+1} + \langle y_{q+1} + z_{q+1}\rangle)) \cap \mathbf{D}(z_{q+1})} \cup (V_m \cap \mathbf{V}(y_{q+1}, z_{q+1})). \end{aligned}$$

Note that $V_m \cap \mathbf{V}(y_{q+1}, z_{q+1}) = W_m$. By Lemma 4.4(a), (b) and (c) and the statement (a) above,

$$\operatorname{codim}_{(\mathbb{A}^3)_m} V_m > m + u + 1.$$

This complete the proof of (b).

(c) Since $(\mathbb{A}^3)_m = \mathbf{D}(y_q) \cup \mathbf{D}(z_q) \cup \mathbf{V}(y_q, z_q)$ and U_m is closed, we have

$$U_m = U_m \cap (\mathbb{A}^3)_m = \overline{U_m \cap \mathbf{D}(y_q)} \cup \overline{U_m \cap \mathbf{D}(z_q)} \cup (U_m \cap \mathbf{V}(y_q, z_q)).$$

By Lemma 4.4(d), we have

$$\overline{U_m \cap \mathbf{D}(y_q)} = \overline{U_m \cap \mathbf{D}(z_q)}$$

and $\overline{U_m \cap \mathbf{D}(y_q)}$ is irreducible of codimension

$$m + p - q + 1 = m + u + 1$$

in $(\mathbb{A}^3)_m$. Moreover, we have $U_m \cap \mathbf{V}(y_q, z_q) = V_m$. By the assertion (b), we have

$$\operatorname{codim}_{(\mathbb{A}^3)_m} V_m > m + u + 1.$$

We note that U_m is defined by the ideal $L_{pqq} + \langle g^{(2p)}_{pqq}, ..., g^{(m)}_{pqq} \rangle$, which is generated by

$$p + q + q + (m - 2p + 1) = m + u + 1$$

elements. Thus, by Krull's height theorem, V_m contains no irreducible component of U_m. Hence

$$U_m = \overline{U_m \cap \mathbf{D}(y_q)}$$

is irreducible of

$$\operatorname{codim}_{(\mathbb{A}^3)_m} U_m = m + u + 1.$$

This complete the proof of (c) and the proof of the proposition. □

Proof of Lemma 4.9. First, we consider the case $m = 5$. Then we have

$$Z^0_m = \mathbf{V}(L_{322})$$

by Remark 4.8, and this closed subset is irreducible of codimension $3 + 2 + 2 = 7 = m + 2$.

Next, we prove the statement for $m \geq 6$. Let u be the positive integer with $6u \leq m < 6(u + 1)$. If $u = 1$, then by Remark 4.8 and Proposition 4.10, $Z^0_m = \mathbf{V}(L_{322} + \langle g^{(0)}, ..., g^{(m)} \rangle) = U_m$ is irreducible of codimension $m + u + 1 = m + 2$. Hence this case was proven. In the following, we assume $u \geq 2$. We note that $\mathbf{V}(L_{322})$ is equal to

$$\bigcup_{u'=2}^{2u-1} \Big(\mathbf{V}(L_{3,u',u'}) \cap (\mathbf{D}(y_{u'}) \cup \mathbf{D}(z_{u'}))\Big) \cup \mathbf{V}(L_{3,2u,2u}).$$

Since Z^0_m can be written as $Z^0_m \cap \mathbf{V}(L_{322})$ by Remark 4.8, it is the union of the following sets:

(α) $Y_{u'} := Z^0_m \cap \Big(\mathbf{V}(L_{3,u',u'}) \cap (\mathbf{D}(y_{u'}) \cup \mathbf{D}(z_{u'}))\Big)$ $(u' = 2, 3, ..., 2u - 1)$,

(β) $Y_{2u} := Z^0_m \cap \mathbf{V}(L_{3,2u,2u})$.

Now, we will see the following claim.

Claim 4.11. (a) *Y_2 is irreducible of codimension $m + 2$ in $(\mathbb{A}^3)_m$.*

(b) *For $u' = 3, 4, ..., 2u$, $Y_{u'}$ is of codimension greater than $m + 2$ in $(\mathbb{A}^3)_m$.*

Note that $L_{322} + \langle g^{(0)}, ..., g^{(m)} \rangle = L_{322} + \langle g^{(6)}_{322}, ..., g^{(m)}_{322} \rangle$ by Remark 2.6 and by Remark 4.8. This ideal is generated by $3 + 2 + 2 + (m - 5) = m + 2$ elements, so we see that any irreducible component of $Z^0_m = \mathbf{V}(I^0_m)$ is of codimension at most $m + 2$ in $(\mathbb{A}^3)_m$. Thus if we can prove the above claim, then the proof is complete.

Before proving the claim, we give another set of defining equations of $Z^0_m \cap \mathbf{V}(L_{3,u',u'})$ for $u' = 2, 3, ..., 2u$. Note that

$$Z^0_m \cap \mathbf{V}(L_{3,u',u'}) = \mathbf{V}(L_{3,u',u'} + \langle g^{(6)}_{3,u',u'}, ..., g^{(m)}_{3,u',u'} \rangle)$$

by Remark 4.8. Let us show that

$$\sqrt{L_{3,u',u'}+\langle g^{(6)}_{3,u',u'},\dots,g^{(m)}_{3,u',u'}\rangle}=\sqrt{L_{\lceil 3u'/2\rceil,u',u'}+\langle g^{(6)}_{3,u',u'},\dots,g^{(m)}_{3,u',u'}\rangle}$$
$$=\sqrt{L_{\lceil 3u'/2\rceil,u',u'}+\langle g^{(6)}_{\lceil 3u'/2\rceil,u',u'},\dots,g^{(m)}_{\lceil 3u'/2\rceil,u',u'}\rangle}. \tag{2}$$

For the case $u'=2$, then $\lceil 3u'/2\rceil=3$ and hence the assertion holds.

For the cases $u'\geq 3$, it is suffices to show that

$$x_{l/2}\in\sqrt{L_{3,u',u'}+\langle g^{(6)}_{3,u',u'},\dots,g^{(m)}_{3,u',u'}\rangle}$$

for $l=6,8,\dots,2(\lceil 3u'/2\rceil-1)$. We prove this by induction. If $l=6$, then $l=6<9\leq 3u'$ $(\leq m)$ and hence $g^{(6)}_{3,u',u'}=x_3^2$ by Lemma 2.14(b), and hence

$$x_3\in\sqrt{L_{3,u',u'}+\langle g^{(6)}_{3,u',u'},\dots,g^{(m)}_{3,u',u'}\rangle}.$$

For $l\geq 8$, we assume that

$$x_{l'/2}\in\sqrt{L_{3,u',u'}+\langle g^{(6)}_{3,u',u'},\dots,g^{(m)}_{3,u',u'}\rangle}$$

holds for even integers $l'<l$. Then we have

$$\sqrt{L_{3,u',u'}+\langle g^{(6)}_{3,u',u'},\dots,g^{(m)}_{3,u',u'}\rangle}=\sqrt{L_{l/2,u',u'}+\langle g^{(6)}_{l/2,u',u'},\dots,g^{(m)}_{l/2,u',u'}\rangle}.$$

Since $l/2<\lceil 3u'/2\rceil<2u'$ for $u'>0$, we have

$$l\leq 2(\lceil 3u'/2\rceil-1)\leq 2((3u'+1)/2-1)=3u'+1-2=3u'-1<3u'\ (\leq m).$$

Hence we have

$$g^{(l)}_{l/2,u',u'}=x_{l/2}^2$$

and

$$x_{l/2}\in\sqrt{L_{l/2,u',u'}+\langle g^{(6)}_{l/2,u',u'},\dots,g^{(m)}_{l/2,u',u'}\rangle}.$$

Therefore, we have

$$x_{l/2}\in\sqrt{L_{3,u',u'}+\langle g^{(6)}_{3,u',u'},\dots,g^{(m)}_{3,u',u'}\rangle}$$

for $l=6,8,\dots,2(\lceil 3u'/2\rceil-1)$ and (2) holds.

Now, we prove the statements (a) and (b) of the claim with $u'\neq 2u$. If u' is even, then, for $u''=u'/2$, we have

$$\overline{Z^0_m\cap\mathbf{V}(L_{3,u',u'})\cap\mathbf{D}(y_{u'})}=\overline{\mathbf{V}(L_{3u'',2u'',2u''}+\langle g^{(6)}_{3u'',2u'',2u''},\dots,g^{(m)}_{3u'',2u'',2u''}\rangle)\cap\mathbf{D}(y_{2u''})},$$
$$\overline{Z^0_m\cap\mathbf{V}(L_{3,u',u'})\cap\mathbf{D}(z_{u'})}=\overline{\mathbf{V}(L_{3u'',2u'',2u''}+\langle g^{(6)}_{3u'',2u'',2u''},\dots,g^{(m)}_{3u'',2u'',2u''}\rangle)\cap\mathbf{D}(z_{2u''})}$$

by (2) and these sets coincide and are irreducible of codimension $m-3u''+2(2u'')+1=m+u''+1$ by Lemma 4.4(d) with $p=3u''$ and $q=r=2u''$. Hence, if $u'=2$, i.e., $u''=1$, then this set is of codimension $m+2$. If $u'\geq 4$ i.e., $u''\geq 2$, then this set of codimension $m+u''+1>m+2$. Thus, the case in which u' is even is complete.

Next, if u' is odd, then we have

$$Z^0_m\cap\mathbf{V}(L_{3,u',u'})=\mathbf{V}(L_{\frac{3u'+1}{2},u',u'}+\langle g^{(6)}_{\frac{3u'+1}{2},u',u'},\dots,g^{(m)}_{\frac{3u'+1}{2},u',u'}\rangle).$$

by (2). By Lemma 2.14(a) and (d) with $p = (3u'+1)/2, q = u'$ and $r = u'$, we have $g^{(l)}_{\frac{3u'+1}{2},u',u'} = 0$ for $l < 3u'$ and $g^{(3u')}_{\frac{3u'+1}{2},u',u'} = y_{u'}z_{u'}(y_{u'} + z_{u'})$. Thus

$$Z^0_m \cap \mathbf{V}(L_{3,u',u'}) = \bigcup_{i=1,2,3} \left(\mathbf{V}(L_{\frac{3u'+1}{2},u',u'} + \langle h_{u',i}\rangle + \langle g^{(3u'+1)}_{\frac{3u'+1}{2},u',u'}, \ldots, g^{(m)}_{\frac{3u'+1}{2},u',u'}\rangle)\right)$$

where $h_{u',1} = y_{u'}$, $h_{u',2} = z_{u'}$ and $h_{u',3} = y_{u'} + z_{u'}$. Therefore,

$$\begin{aligned}&\overline{Z^0_m \cap \mathbf{V}(L_{3,u',u'}) \cap \mathbf{D}(y_{u'})}\\ &= \bigcup_{i=2,3} \overline{\mathbf{V}(L_{\frac{3u'+1}{2},u',u'} + \langle h_{u',i}\rangle + \langle g^{(3u'+1)}_{\frac{3u'+1}{2},u',u'}, \ldots, g^{(m)}_{\frac{3u'+1}{2},u',u'}\rangle) \cap \mathbf{D}(y_{u'})}\end{aligned}$$

and both summands are irreducible of codimension $m + p - q + 1 > m + 2$ by Lemma 4.4(b) with $p = (3u'+1)/2$, $q = u'$ and $r = u'+1$ and Lemma 4.4(c) with $p = (3u'+1)/2$ and $q = r = u'$. By the symmetry, we have the same conclusion for

$$\overline{Z^0_m \cap \mathbf{V}(L_{3,u',u'}) \cap \mathbf{D}(z_{u'})}.$$

Hence, for $2 < u' \leq 2u-1$, $Y_{u'}$ is of codimension greater than $m+2$. Thus, the case in which u' is odd is complete.

Finally, we prove (b) with $u' = 2u$. By (2),

$$Y_{2u} = Z^0_m \cap \mathbf{V}(L_{3,2u,2u}) = \mathbf{V}(L_{3u,2u,2u} + \langle g^{(6)}_{3u,2u,2u}, \ldots, g^{(m)}_{3u,2u,2u}\rangle).$$

Then this closed subset is irreducible of codimension $m + u + 1$ by Proposition 4.10. Furthermore, we have $u > 1$ by assumption. Hence Y_{2u} is of codimension $m + u + 1 > m + 2$. □

We remark on the symmetries of the irreducible components of the singular fiber.

Lemma 4.12. *Assume $m \geq 3$. The closed subsets Z^0_m, Z^1_m, Z^2_m and Z^3_m are mapped to another by ψ_1 and ψ_2 (see Notation 4.2) as follows:*

(a) $\psi_1(Z^0_m) = Z^0_m$, $\psi_1(Z^1_m) = Z^2_m$, $\psi_1(Z^2_m) = Z^1_m$ *and* $\psi_1(Z^3_m) = Z^3_m$.
(b) $\psi_2(Z^0_m) = Z^0_m$, $\psi_2(Z^1_m) = Z^1_m$, $\psi_2(Z^2_m) = Z^3_m$ *and* $\psi_2(Z^3_m) = Z^2_m$.

By Lemma 4.3, we can prove this lemma in the same way as Lemma 3.8.

Proposition 4.13. *For $m \geq 3$, we have*

$$S^0_m = Z^0_m \cup Z^1_m \cup Z^2_m \cup Z^3_m.$$

Moreover, Z^1_m, Z^2_m and Z^3_m are pairwise distinct.

By Remark 2.15, the assertions in Lemma 2.14 (a) – (h) are the same as f and g. Hence the proof of this proposition is the same as that of Proposition 3.9.

Now, we give the irreducible decomposition of the singular fiber S^0_m. The following proposition gives the irreducible decomposition for small m.

Proposition 4.14. *For $0 \leq m \leq 4$, the irreducible decomposition of the singular fiber S^0_m is as follows.*

(a) $S^0_0 = \mathbf{V}(L_{111})$,
(b) $S^0_1 = \mathbf{V}(L_{111})$,
(c) $S^0_2 = \mathbf{V}(L_{211})$,
(d) $S^0_3 = Z^1_3 \cup Z^2_3 \cup Z^3_3$,
(e) $S^0_4 = Z^1_4 \cup Z^2_4 \cup Z^3_4$,

where Z^i_m's are as in Definition 4.7. Moreover, for $1 \leq m \leq 4$, the codimension of any irreducible component $Z \subseteq S^0_m$ is

$$\operatorname{codim}_{(\mathbb{A}^3)_m} Z = m + 2.$$

This proposition can be proved in the same way as Proposition 3.10.

Theorem 4.15. *For $m \geq 5$, the irreducible decomposition of the singular fiber S_m^0 is*

$$S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3,$$

where Z_m^i's are as in Definition 4.7.

Proof. By Proposition 4.13,

$$S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3.$$

Moreover, by Corollary 4.6 and Lemma 4.9, Z_m^0, Z_m^1, Z_m^2 and Z_m^3 are irreducible of codimensions $m+2$ in $(\mathbb{A}^3)_m$. Hence we only have to show that Z_m^i, for $0 \leq i \leq 3$, are pairwise distinct.

For $m \geq 5$, we have $Z_m^0 \cap (\mathbf{D}(y_1) \cup \mathbf{D}(z_1)) = \emptyset$ while $Z_m^i \cap (\mathbf{D}(y_1) \cup \mathbf{D}(z_1)) \neq \emptyset$ for $1 \leq i \leq 3$, so Z_m^0 is different from Z_m^1, Z_m^2 and Z_m^3. Moreover, by Proposition 4.13 Z_m^1, Z_m^2 and Z_m^3 are pairwise distinct. This complete the proof. $\square$

To conclude this paper, we prove the following theorem on the jet schemes of D_4^1-singularities, which is analogous to Theorem 3.15 and the characteristic 0 case [5, Theorem 3.17].

Theorem 4.16. *Let k be an algebraically closed field of characteristic 2, $S \subset \mathbb{A}^3$ the surface defined by $g = x^2 + y^2z + yz^2 + xyz$ in the affine space over k, S_m^0 the singular fiber of the m-th jet scheme S_m with $m \geq 5$ and $Z_m^0, ..., Z_m^3$ its irreducible components as in Definition 4.7.*

(a) *For $0 \leq i < j \leq 3$, $Z_m^i \cap Z_m^j \subsetneq Z_m^0$.*
(b) *For $1 \leq i, j \leq 3$ with $i \neq j$, $Z_m^0 \cap Z_m^i \not\subseteq Z_m^0 \cap Z_m^j$.*
(c) *For $1 \leq i, j \leq 3$ with $i \neq j$, $Z_m^i \cap Z_m^j \subsetneq Z_m^0 \cap Z_m^i$.*
(d) *For $1 \leq i < j \leq 3$ and $1 \leq l \leq 3$, $Z_m^0 \cap Z_m^l \not\subseteq Z_m^i \cap Z_m^j$.*

In particular, for $0 \leq i < j \leq 3$, $Z_m^i \cap Z_m^j$ is maximal in $\{Z_m^i \cap Z_m^j \mid i, j \in \{0, 1, 2, 3\}, i \neq j\}$ with respect to the inclusion relation if and only if $(i, j) = (0, 1), (0, 2), (0, 3)$.

Proof. We argue as in the proof of Theorem 3.15.

(a) Note that

$$g^{(4)} \equiv f^{(4)}$$

modulo L_{221} or L_{212} for $f = x^2 + y^2z + yz^2$ since terms from $\mathbf{x}_2\mathbf{y}_2\mathbf{z}_1$ or $\mathbf{x}_2\mathbf{y}_1\mathbf{z}_2$ are of degree at least 5, where $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i$ are as in Notation 2.5. Hence we can prove $Z_m^i \cap Z_m^j \subsetneq Z_m^0$ as in the proof of Theorem 3.15.

(b) By Lemma 4.12, it suffices to show that

(i) $(0, t^2, t^2) \in Z_m^0 \cap Z_m^3$.
(ii) $(0, t^2, t^2) \notin Z_m^0 \cap Z_m^1$.

Let $P = (0, t^2, t^2)$. We have $g(P) = 0^2 + (t^2)^2 \cdot t^2 + t^2 \cdot (t^2)^2 + 0 \cdot t^2 \cdot t^2 = 2t^6 = 0$, hence we have $P \in S_m$.

(i) Let us prove $P \in Z_m^0 \cap Z_m^3$. We note that P corresponds to $x_\alpha = y_\beta = z_\beta = 0$ for $\alpha \in \{0, ..., m\}$ and $\beta \in \{0, 1, 3, ..., m\}$ and $y_2 = z_2 = 1$. Thus we have $P \in Z_m^0 = S_m \cap \mathbf{V}(L_{222})$. We put $P_s = (0, st + t^2, st + t^2)$ for $s \in k$. If $s \neq 0$, then we have $g(P_s) = 0$ and P_s corresponds to $x_\alpha = y_\beta = z_\beta = 0$ for $\alpha \in \{0, ..., m\}$ and $\beta \in \{0, 3, ..., m\}$, $y_1 = z_1 = s \neq 0$ and $y_2 = z_2 = 1$. Hence we have $P_s \in \mathbf{V}(L_{211} + \langle y_1 + z_1 \rangle) \cap \mathbf{D}(y_1)$ for $s \neq 0$. Taking the Zariski closure of $\mathbf{V}(L_{211} + \langle y_1 + z_1 \rangle) \cap \mathbf{D}(y_1)$, we have $P = P_0 \in \overline{\mathbf{V}(L_{211} + \langle y_1 + z_1 \rangle) \cap \mathbf{D}(y_1)} = Z_m^3$. Therefore, we have $P \in Z_m^0 \cap Z_m^3$.

(ii) We prove $P \notin Z_m^0 \cap Z_m^1$. We have

$$g_{221}^{(5)} = y_2^2 z_1 + y_3 z_1^2 + x_2 y_2 z_1 = z_1(y_2^2 + y_3 z_1 + x_2 y_2),$$

hence we have $y_2^2 + y_3z_1 + x_2y_2 \in J_m^1 \cdot (R_m)_{z_1} \cap R_m = I_m^1$. Since $x_2, z_1 \in L_{322} \subseteq \sqrt{I_m^0}$ (see Remark 4.8), we have $y_2 \in \sqrt{I_m^0 + I_m^1}$. Therefore we have $P \notin Z_m^0 \cap Z_m^1$.

(c), (d) The assertions are proven in the same way as in the proof of Theorem 3.15(c) and (d). □

Corollary 4.17. *For $m \geq 5$, the graph $\Gamma(S_m^0)$ obtained by Construction 3.17 is the resolution graph of a D_4-type singularity.*

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