

## HIGHER JACOBIAN MATRIX OF WEIGHTED HOMOGENEOUS POLYNOMIALS AND DERIVATION ALGEBRAS

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ABSTRACT. We prove that the ideal generated by the maximal minors of the higher-order Jacobian matrix of a weighted homogeneous polynomial is also weighted homogeneous. As an application, we give a partial answer to a conjecture concerning the non-existence of negative weight derivations on the higher Nash blowup local algebra of a hypersurface.

### INTRODUCTION

The Jacobian matrix of order  $n$  is a higher-order version of the classical Jacobian matrix. It was introduced as a tool for computing the higher Nash blowup of a hypersurface [Dua17]. Higher-order Jacobian matrices were later rediscovered and further developed by several authors [BJNB19, BD20, CMDGF21]. Ever since, this matrix has seen a wide variety of applications: in the study of higher Nash blowups [Dua17, CMDGF21], the study of invariants of rings in positive characteristic [BJNB19], the study of homological properties of the module of higher Kähler differentials [BD20, dAD21], relations with Hasse-Schmidt derivations and jet schemes [LDS24, Bar23], invariants of germs of hypersurface singularities [HMYZ23, LY24], motivic zeta functions [LY24]. In addition to these applications, new questions have been raised regarding general properties of higher Jacobian matrices.

Let  $f \in \mathbb{C}\{x_1, \dots, x_s\}$ , where  $\mathbb{C}\{x_1, \dots, x_s\}$  denotes the ring of convergent power series at the origin. Let  $\text{Jac}_n(f)$  be the higher Jacobian matrix of  $f$  and  $\mathcal{J}_n(f)$  the ideal generated by the maximal minors of  $\text{Jac}_n(f)$  (see Definition 1.3). N. Hussain, G. Ma, S. S.-T. Yau, and H. Zuo defined the higher Nash blowup local algebra as the quotient  $\mathcal{T}_n(f) = \mathbb{C}\{x_1, \dots, x_s\} / \langle f, \mathcal{J}_n(f) \rangle$  [HMYZ23]. For  $n = 1$ , this is the classical Tjurina algebra. The authors proposed several conjectures regarding  $\mathcal{T}_n(f)$ : invariance under contact equivalence, homogeneity properties, and bounds for their dimensions. They also provided some evidence that supports the conjectures. These questions can be seen as higher-order analogues of classical results in singularity theory. For instance, a famous result by J. Mather and S. S.-T. Yau states that the Tjurina algebra is a complete invariant of isolated hypersurface singularities under contact equivalence [MY82]. Moreover, higher-order versions of the Tjurina algebra have been an object of intense study (see, for instance, [GLS07, DS15, GP17, DGI20, MYZ23]).

Very recently, the conjecture regarding the invariance of  $\mathcal{T}_n(f)$  under contact equivalence was proved by Q. T. Lê and T. Yasuda [LY24]. In this paper we are interested in exploring the questions regarding the homogeneity properties of the higher Jacobian matrix. The following is our first main theorem.

**Theorem A** (see Theorem 1.7). *Let  $f \in \mathbb{C}[x_1, \dots, x_s]$  be a weighted homogeneous polynomial. Then  $\mathcal{J}_n(f)$  is a weighted homogeneous ideal for every  $n \in \mathbb{N}$ .*

Another object introduced by the aforementioned authors is the algebra of derivations  $\mathcal{L}_n(f) = \text{Der}(\mathcal{T}_n(f))$ . For  $n = 1$ , this is known as the Yau algebra of  $f$ . A classical result

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in singularity theory states that the Yau algebra is a solvable Lie algebra for isolated hypersurface singularities [Yau86, Yau91]. Moreover, the Yau algebra has been used to distinguish complex analytic structures of isolated hypersurface singularities [SY90].

By Theorem A,  $\mathbb{C}[x_1, \dots, x_s]/\langle f, \mathcal{J}_n(f) \rangle$  is a graded algebra whenever  $f$  is a weighted homogeneous polynomial. In this case, the algebra  $\text{Der}(\mathbb{C}[x_1, \dots, x_s]/\langle f, \mathcal{J}_n(f) \rangle)$  is also graded. There is a general conjecture on algebras of derivations of local Artinian graded algebras, known as Halperin conjecture, stating the non-existence of negative weight derivations over those algebras [Che99]. This conjecture has been intensively studied [Mei82, FHT01, CYZ19, CCYZ20, CHYZ20]. Inspired by Halperin’s question, N. Hussain, G. Ma, S. S.-T. Yau, and H. Zuo conjectured that there are no negative weight derivations on  $\mathbb{C}[x_1, \dots, x_s]/\langle f, \mathcal{J}_n(f) \rangle$ . They proved their conjecture in the case  $n = s = 2$ . Our second main result shows that the conjecture is true for  $n \geq 3$  and  $s = 2$ .

**Theorem B** (see Theorem 2.6). *Let  $f \in \mathbb{C}[x_1, x_2]$  be a weighted homogeneous polynomial of weight  $(w_1, w_2) \in \mathbb{N}^2$  and degree  $d$  that defines an isolated hypersurface singularity. Suppose that  $d \geq 2w_1 \geq 2w_2 > 0$ . Let  $n \geq 3$ . Then  $\text{Der}(\mathbb{C}[x_1, x_2]/\langle f, \mathcal{J}_n(f) \rangle)$  is non-negatively graded.*

This paper is divided as follows. In the first section we prove Theorem A. We also establish some facts regarding the weighted degree of the maximal minors of the higher Jacobian matrix. The second section contains the proof of Theorem B.

1. HIGHER-ORDER JACOBIAN MATRIX OF A WEIGHTED HOMOGENEOUS POLYNOMIAL

The following notation will be constantly used throughout this paper.

**Notation 1.1.** Let  $n, s \in \mathbb{N}_{\geq 1}$ . Given  $\gamma = (\gamma_1, \dots, \gamma_s) \in \mathbb{N}^s$ , denote  $|\gamma| = \gamma_1 + \dots + \gamma_s$ . For  $\gamma, \gamma' \in \mathbb{N}^s$  we denote  $\gamma \cdot \gamma'$  the usual Euclidean inner product. In addition, consider

$$\begin{aligned} B &:= \{\beta \in \mathbb{N}^s \mid 0 \leq |\beta| \leq n - 1\}, \\ B_1 &:= \{\beta \in \mathbb{N}^s \mid 1 \leq |\beta| \leq n - 1\}, \\ A &:= \{\alpha \in \mathbb{N}^s \mid 0 \leq |\alpha| \leq n\}, \\ A_1 &:= \{\alpha \in \mathbb{N}^s \mid 1 \leq |\alpha| \leq n\}, \\ C_n &:= \{\alpha \in \mathbb{N}^s \mid |\alpha| = n\}. \end{aligned}$$

**Remark 1.2.** It is known that

$$\begin{aligned} \text{Card}(B) &= \binom{s+n-1}{s} =: M, \\ \text{Card}(A) &= \binom{s+n}{s} =: N, \\ \text{Card}(C_n) &= \binom{n+s-1}{s-1} = N - M =: l. \end{aligned}$$

**Definition 1.3** ([Dua17, BJNB19, BD20]). Let  $f \in \mathbb{C}\{x_1, \dots, x_s\}$ . Denote

$$\text{Jac}_n(f) := \left( \frac{1}{(\alpha - \beta)!} \frac{\partial^{\alpha - \beta}(f)}{\partial x^{\alpha - \beta}} \right)_{\substack{\beta \in B \\ \alpha \in A_1}},$$

where we define  $\frac{1}{(\alpha - \beta)!} \frac{\partial^{\alpha - \beta}(f)}{\partial x^{\alpha - \beta}} = 0$ , whenever  $\alpha_i < \beta_i$  for some  $i$ . It is a  $M \times (N - 1)$ -matrix. We call  $\text{Jac}_n(f)$  the higher-order Jacobian matrix of  $f$  or the Jacobian matrix of order  $n$  of  $f$ . We order this matrix increasingly using a graded lexicographical order and taking  $x_1 < x_2 < \dots < x_s$ . Moreover, denote as  $\mathcal{J}_n(f)$  the ideal generated by all maximal minors of  $\text{Jac}_n(f)$ .

**Remark 1.4.** The higher-order Jacobian matrix was originally introduced as a tool to compute the higher Nash blowup of a hypersurface [Yas07, Dua17].

**Example 1.5.** Let  $f = x^3 - y^2 \in \mathbb{C}[x, y]$ . Then

$$\text{Jac}_2(f) = \begin{pmatrix} 3x^2 & -2y & 3x & 0 & -1 \\ f & 0 & 3x^2 & -2y & 0 \\ 0 & f & 0 & 3x^2 & -2y \end{pmatrix}.$$

N. Hussain, G. Ma, S. S.-T. Yau, and H. Zuo proposed the following conjecture regarding homogeneity properties of  $\text{Jac}_n(f)$ .

**Conjecture 1.6** ([HMYZ23, Conjecture 1.7]). Let  $f \in \mathbb{C}[x_1, \dots, x_s]$  be a weighted homogeneous polynomial. Then  $\mathcal{J}_n(f)$  is a weighted homogeneous ideal.

The aforementioned authors verified Conjecture 1.6 in the case  $s = n = 2$  [HMYZ23, Lemma 4.1]. The following theorem gives a positive answer to Conjecture 1.6 for arbitrary  $n$  and  $s$ .

**Theorem 1.7.** Let  $f$  be a weighted homogeneous polynomial in  $\mathbb{C}[x_1, \dots, x_s]$  of weight  $w \in \mathbb{N}^s$  and degree  $d$ . Then  $\mathcal{J}_n(f)$  is a weighted homogeneous ideal for every  $n \in \mathbb{N}$ .

*Proof.* Let  $L$  be a matrix formed by taking  $M$  columns of  $\text{Jac}_n(f)$ . Let  $B = \{\beta(1), \dots, \beta(M)\}$  and  $\{\alpha(1), \dots, \alpha(M)\} \subset A_1$  be the sets of vectors indexing the rows and columns of  $L$ , respectively. Denote the entries of  $L$  as  $L_{\beta(i)\alpha(j)}$  with  $1 \leq i, j \leq M$ . Recall that

$$L_{\beta(i)\alpha(j)} = \frac{1}{(\alpha(j) - \beta(i))!} \frac{\partial^{\alpha(j) - \beta(i)}(f)}{\partial x^{\alpha(j) - \beta(i)}}.$$

Set  $I = \{\theta \in S_M \mid L_{\beta(i)\alpha(\theta(i))} \neq 0 \text{ for every } 1 \leq i \leq M\}$ , where  $S_M$  denotes the symmetric group of  $M$  elements. We have that

$$\det(L) = \sum_{\theta \in I} \text{sgn}(\theta) L_{\beta(1)\alpha(\theta(1))} \cdots L_{\beta(M)\alpha(\theta(M))}.$$

Notice that  $f$  being weighted homogeneous of degree  $d$  implies that  $L_{\beta(i)\alpha(\theta(i))}$  is a weighted homogeneous polynomial of degree  $d - (\alpha(\theta(i)) - \beta(i)) \cdot w$ . Hence,  $L_{\beta(1)\alpha(\theta(1))} \cdots L_{\beta(M)\alpha(\theta(M))}$  is a weighted homogeneous polynomial of degree  $dM - \sum_{i=1}^M (\alpha(\theta(i)) - \beta(i)) \cdot w$ .

For every  $\sigma \in I$  we claim that

$$\sum_{i=1}^M (\alpha(\sigma(i)) - \beta(i)) \cdot w = \sum_{i=1}^M (\alpha(i) - \beta(i)) \cdot w.$$

Indeed,

$$\begin{aligned} 0 &= \sum_{i=1}^M (\alpha(\sigma(i)) - \alpha(i)) \cdot w \\ &= \sum_{i=1}^M (\alpha(\sigma(i)) - \beta(i) + \beta(i) - \alpha(i)) \cdot w \\ &= \sum_{i=1}^M (\alpha(\sigma(i)) - \beta(i)) \cdot w - \sum_{i=1}^M (\alpha(i) - \beta(i)) \cdot w. \end{aligned}$$

Taking  $c = \sum_{i=1}^M (\alpha(i) - \beta(i)) \cdot w$ , we conclude  $L_{\beta(1)\alpha(\theta(1))} \cdots L_{\beta(M)\alpha(\theta(M))}$  is a weighted homogeneous polynomial of weight  $w$  and degree  $dM - c$  for every  $\theta \in I$ . Therefore,  $\det(L)$  is a weighted homogeneous polynomial of weight  $w$  and degree  $dM - c$ .  $\square$

Notice that the proof of the previous theorem also exhibited the degree of the maximal minors of  $\text{Jac}_n(f)$ . Our next goal is to give lower bounds for those degrees, in some special cases.

**1.1. Homogeneous case.** Let  $f$  be a homogeneous polynomial in  $\mathbb{C}[x_1, \dots, x_s]$  of degree  $d$ , and let  $n \geq 2$ ,  $s \geq 2$ . Let  $g$  be the determinant of the submatrix of  $\text{Jac}_n(f)$  formed by the columns indexed by  $\{\alpha(1), \dots, \alpha(M)\} \subset A_1$  and the rows indexed by  $B = \{\beta(1), \dots, \beta(M)\}$ . From the proof of Theorem 1.7,  $g$  is a homogeneous polynomial of degree  $dM - c$ , where

$$c = \sum_{i=1}^M |\alpha(i) - \beta(i)| = \sum_{i=1}^M |\alpha(i)| - \sum_{i=1}^M |\beta(i)|.$$

**Lemma 1.8.** *With the previous notation we have  $sM \geq c + s$ .*

*Proof.* We can assume that  $1 \leq |\alpha(1)| \leq \dots \leq |\alpha(M)| \leq n$ . Recall the notation from Remark 1.2 and denote  $\mathcal{A} := \{\alpha(1), \dots, \alpha(M-l)\}$ . Notice that  $\mathcal{A} \subset B_1$ , thus

$$\sum_{i=1}^M |\alpha(i)| \leq \sum_{\alpha \in \mathcal{A}} |\alpha| + \sum_{\alpha \in C_n} |\alpha| = \sum_{\alpha \in \mathcal{A}} |\alpha| + ln = \sum_{\alpha \in \mathcal{A}} |\alpha| + sM.$$

On the other hand,  $\text{Card}(B_1 \setminus \mathcal{A}) = (M-1) - (M-l) = l-1 \geq s$ . This implies

$$\sum_{i=1}^M |\beta(i)| = \sum_{\beta \in B_1} |\beta| \geq \sum_{\alpha \in \mathcal{A}} |\alpha| + s.$$

From the above,

$$\begin{aligned} c + s &= \sum_{i=1}^M |\alpha(i)| - \sum_{i=1}^M |\beta(i)| + s \\ &\leq sM + \sum_{\alpha \in \mathcal{A}} |\alpha| - \sum_{i=1}^M |\beta(i)| + s \\ &\leq sM. \end{aligned}$$

□

**Corollary 1.9.** *Let  $f \in \mathbb{C}[x_1, \dots, x_s]$  be a homogeneous polynomial of degree  $d \geq s$  and let  $n \geq 2$ . Then  $d$  is a lower bound for the degrees of the maximal minors of  $\text{Jac}_n(f)$ . In other words,  $d \geq s$  implies  $dM - c \geq d$ .*

*Proof.* By Lemma 1.8,  $d(M-1) - c \geq s(M-1) - c \geq 0$ . □

**1.2. Weighted homogeneous case.** In this subsection we work with weighted homogeneous polynomials assuming  $s = 2$  and  $n \geq 3$ . Recall Notation 1.1. In this case we have

$$C_n = \{(n-i, i) \mid 0 \leq i \leq n\}.$$

Notice that  $\text{Card}(C_n) = n+1$  and  $M = \frac{n(n+1)}{2}$ .

As in the previous subsection, we want to give a lower bound for the weighted degrees of the maximal minors of  $\text{Jac}_n(f)$ . Because we are working with an arbitrary weight, the strategy for finding such bound is not as straightforward as the ones of Lemma 1.8 and Corollary 1.9.

**Lemma 1.10.** *Let  $w = (w_1, w_2) \in \mathbb{N}^2$ . Then  $\sum_{\alpha \in C_n} \alpha \cdot w = |w|M$ .*

*Proof.* We have that

$$\begin{aligned} \sum_{\alpha \in C_n} \alpha \cdot w &= \sum_{i=0}^n (n-i, i) \cdot (w_1, w_2) \\ &= \sum_{i=0}^n (n-i)w_1 + \sum_{i=0}^n iw_2 \end{aligned}$$

$$\begin{aligned}
&= \frac{n(n+1)}{2}w_1 + \frac{n(n+1)}{2}w_2 \\
&= |w|M.
\end{aligned}$$

□

Before going any further, we present an example that illustrates the notation and the key idea we develop to find the desired bound.

**Example 1.11.** Let  $n = 3$ . Thus  $M = 6$ . In this case,

$$\begin{aligned}
B &= \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\}, \\
B_1 &= \{(1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\}, \\
A_1 &= \{(1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (3, 0), (2, 1), (1, 2), (0, 3)\}, \\
C_3 &= \{(3, 0), (2, 1), (1, 2), (0, 3)\}.
\end{aligned}$$

Consider the submatrix of  $\text{Jac}_3(f)$  defined by the columns

$$\mathfrak{A} = \{(1, 0), (0, 1), (2, 0), (1, 1), (3, 0), (1, 2)\}.$$

Let us show how we bound the sums  $\sum_{\alpha \in \mathfrak{A}} \alpha \cdot w$  and  $\sum_{\beta \in B} \beta \cdot w$ .

Let  $C = \mathfrak{A} \setminus C_3 = \{(1, 0), (0, 1), (2, 0), (1, 1)\}$ . Let  $\theta = (1, 0)$  and  $\tau = (2, 1)$ . Notice that  $\theta \in C$ ,  $\tau \in C_3 \setminus \mathfrak{A}$ , and  $\theta \cdot w < \tau \cdot w$ . This implies

$$\begin{aligned}
\sum_{\alpha \in \mathfrak{A}} \alpha \cdot w &= \sum_{\alpha \in C} \alpha \cdot w + \sum_{\alpha \in \mathfrak{A} \cap C_3} \alpha \cdot w \\
&= \sum_{\alpha \in C \setminus \{\theta\}} \alpha \cdot w + \theta \cdot w + \sum_{\alpha \in \mathfrak{A} \cap C_3} \alpha \cdot w \\
&< \sum_{\alpha \in C \setminus \{\theta\}} \alpha \cdot w + \tau \cdot w + \sum_{\alpha \in \mathfrak{A} \cap C_3} \alpha \cdot w \\
&< \sum_{\alpha \in C \setminus \{\theta\}} \alpha \cdot w + \sum_{\alpha \in C_3} \alpha \cdot w \\
&= \sum_{\alpha \in C \setminus \{\theta\}} \alpha \cdot w + |w|M.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\sum_{\beta \in B} \beta \cdot w &= \sum_{\beta \in B_1} \beta \cdot w = \left( \sum_{\substack{\beta \in B_1 \\ \beta \neq (1,0), (0,2)}} \beta \cdot w \right) + (1, 0) \cdot w + (0, 2) \cdot w \\
&= \left( \sum_{\alpha \in C \setminus \{\theta\}} \alpha \cdot w \right) + (1, 0) \cdot w + (0, 2) \cdot w \\
&> \sum_{\alpha \in C \setminus \{\theta\}} \alpha \cdot w + w_1 + w_2.
\end{aligned}$$

The following lemma generalizes the process of the previous example.

**Lemma 1.12.** Let  $w = (w_1, w_2) \in \mathbb{N}^2$ . Assume  $w_1 \geq w_2 > 0$ . Let

$$\mathfrak{A} = \{\alpha(1), \dots, \alpha(M)\} \subset A_1, \quad B = \{\beta(1), \dots, \beta(M)\}, \quad \text{and } C = \mathfrak{A} \setminus C_n.$$

Define subsets  $\mathcal{A} \subset C$  satisfying the following conditions:

- (1)  $\mathcal{A} = C$ ,  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$  and there exists  $\theta \in B_1 \setminus \mathcal{A}$  such that  $\theta_1 \geq 1$ .

- (2)  $\mathcal{A} = C \setminus \{\theta\}$ , where  $\theta \in C$  is such that  $\theta_1 \geq 1$ ,  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$ , and there exists  $\tau \in C_n \setminus \mathfrak{A}$  such that  $\theta \cdot w \leq \tau \cdot w$ .
- (3)  $\mathcal{A} = C \setminus \{\theta, \theta'\}$ , where  $\theta, \theta' \in C$  are such that  $\theta_1 \geq 1$  and there exist  $\tau, \tau' \in C_n \setminus \mathfrak{A}$  such that  $\theta \cdot w \leq \tau \cdot w$  and  $\theta' \cdot w \leq \tau' \cdot w$ .

Then  $|w|M \geq c + |w|$ , where  $c = \sum_{i=1}^M (\alpha(i) - \beta(i)) \cdot w$ .

*Proof.* In each of the cases we have:

$$\begin{aligned}
\sum_{i=1}^M \alpha(i) \cdot w &= \sum_{\alpha \in C} \alpha \cdot w + \sum_{\alpha \in \mathfrak{A} \cap C_n} \alpha \cdot w \\
&= \sum_{\alpha \in \mathcal{A}} \alpha \cdot w + \sum_{\alpha \in C \setminus \mathcal{A}} \alpha \cdot w + \sum_{\alpha \in \mathfrak{A} \cap C_n} \alpha \cdot w \\
&\leq \sum_{\alpha \in \mathcal{A}} \alpha \cdot w + \sum_{\alpha \in C_n} \alpha \cdot w \\
&= \sum_{\alpha \in \mathcal{A}} \alpha \cdot w + |w|M,
\end{aligned}$$

where the last equality follows by Lemma 1.10.

On the other hand, assuming that  $\beta(1) = (0, 0)$ , since  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$  we also have:

$$\sum_{i=1}^M \beta(i) \cdot w = \sum_{i=2}^M \beta(i) \cdot w \geq \sum_{\alpha \in \mathcal{A}} \alpha \cdot w + (w_1 + w_2).$$

Finally,

$$\begin{aligned}
c &= \sum_{i=1}^M \alpha(i) \cdot w - \sum_{i=1}^M \beta(i) \cdot w \\
&\leq |w|M + \sum_{\alpha \in \mathcal{A}} \alpha \cdot w - \sum_{i=1}^M \beta(i) \cdot w \\
&\leq |w|M + \sum_{\alpha \in \mathcal{A}} \alpha \cdot w - \sum_{\alpha \in \mathcal{A}} \alpha \cdot w - (w_1 + w_2) \\
&= |w|M - |w|.
\end{aligned}$$

□

**Remark 1.13.** Let  $f$  be a weighted homogeneous polynomial in  $\mathbb{C}[x_1, x_2]$  of weight  $w \in \mathbb{N}^2$  and degree  $d$ . Without loss of generality we can assume that  $w_1 \geq w_2 > 0$ . Let  $n \geq 3$  and  $g$  be the determinant of the submatrix of  $\text{Jac}_n(f)$  formed by the columns indexed by  $\mathfrak{A} = \{\alpha(1), \dots, \alpha(M)\} \subset A_1$  and the rows indexed by  $B = \{\beta(1), \dots, \beta(M)\}$ . From Theorem 1.7,  $g$  is a weighted homogeneous polynomial of weight  $w$  and degree  $dM - c$ , where

$$\begin{aligned}
c &= \sum_{i=1}^M (\alpha(i) - \beta(i)) \cdot w \\
&= \sum_{i=1}^M \alpha(i) \cdot w - \sum_{i=1}^M \beta(i) \cdot w.
\end{aligned}$$

We take  $C = \mathfrak{A} \setminus C_n \subset B_1$ . We note that  $\mathfrak{A} \cap C_n \neq \emptyset$  since  $\text{Card}(B_1) = M - 1$ . Hence  $M - (n + 1) \leq \text{Card}(C) \leq M - 1$ . Furthermore, denote  $I_j = \{(0, 1), (0, 2), \dots, (0, j)\}$  for  $1 \leq j \leq n$ , and  $C_i = \{\beta \in \mathbb{N}^2 \mid |\beta| = i\}$ , for  $0 \leq i \leq n$ .

**Proposition 1.14.** *Consider the assumptions of Remark 1.13. If  $\text{Card}(C) = M - n$ , then the following cases hold.*

- (1) *If  $\mathfrak{A} = (B_1 \cup C_n) \setminus I_n$ , then  $c = w_1 M$ .*
- (2) *If  $\mathfrak{A} \neq (B_1 \cup C_n) \setminus I_n$ , then  $|w|M \geq c + |w|$ .*

*Proof.* In order to show (1), we first notice

$$\begin{aligned} \sum_{i=1}^M \beta(i) &= \sum_{i=0}^{n-1} \left( \sum_{\beta \in C_i} \beta \right) \\ &= \sum_{i=0}^{n-1} \left( \frac{i(i+1)}{2}, \frac{i(i+1)}{2} \right) \\ &= \sum_{i=1}^n \left( \frac{(i-1)i}{2}, \frac{(i-1)i}{2} \right). \end{aligned}$$

Besides, we note that

$$\begin{aligned} \sum_{i=1}^M \alpha(i) &= \sum_{i=1}^n \left( \sum_{\substack{\alpha \in C_i \\ \alpha \neq (0,i)}} \alpha \right) \\ &= \sum_{i=1}^n \left( \frac{i(i+1)}{2}, \frac{i(i-1)}{2} \right). \end{aligned}$$

Therefore,  $c = \sum_{i=1}^M (\alpha(i) - \beta(i)) \cdot w = \sum_{i=1}^n (i, 0) \cdot w = w_1 M$ .

For (2) we consider two cases:

- $C \neq B_1 \setminus I_{n-1}$ . In this case, there exists  $\theta \in B_1 \setminus C$  such that  $\theta_1 \geq 1$ . Set  $\mathcal{A} = C$  and so  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$ . Then  $|w|M \geq c + |w|$  by Lemma 1.12 (1).
- $C = B_1 \setminus I_{n-1}$ . In this case  $\theta = (1, 0) \in C$ , and  $C_n \setminus \mathfrak{A}$  has a unique element  $\tau \neq (0, n)$ . Then,  $\theta \cdot w \leq \tau \cdot w$ . Set  $\mathcal{A} = C \setminus \{\theta\}$  and so  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$ . Then  $|w|M \geq c + |w|$  by Lemma 1.12 (2).

□

**Proposition 1.15.** *Consider the assumptions of Remark 1.13. If  $\text{Card}(C) \neq M - n$ , then  $|w|M \geq c + |w|$ .*

*Proof.* We proceed by analyzing all possible cases we can have. Assume  $B_1 = B \setminus \{\beta(1)\}$ .

- (1)  $\text{Card}(C) = M - 1$ . In this case we have  $C = B_1$ . We have that  $\theta = (1, 0), \theta' = (0, 1) \in C$ . Since  $\text{Card}(C_n \setminus \mathfrak{A}) = n \geq 3$ , there exist two vectors  $\tau, \tau' \in C_n \setminus \mathfrak{A}$ ,  $\tau \neq \tau'$  such that  $\tau_1, \tau'_2 \geq 1$ , so that  $\theta \cdot w \leq \tau \cdot w$  and  $\theta' \cdot w \leq \tau' \cdot w$ . In this case we take  $\mathcal{A} = C \setminus \{\theta, \theta'\}$  and the result follows by Lemma 1.12(3).
- (2)  $\text{Card}(C) = M - 2$ . In this case  $C = \mathfrak{A} \setminus C_n = B_1 \setminus \{\beta(j)\}$  for some  $j \geq 2$ . We have two subcases:
  - (a)  $\theta = (1, 0) \in C$ . In this case, since  $\text{Card}(C_n \setminus \mathfrak{A}) = n - 1 \geq 2$ , there exists  $\tau \in C_n \setminus \mathfrak{A}$ , such that  $\tau_1 \geq 1$  and  $\theta \cdot w \leq \tau \cdot w$ . We can take  $\mathcal{A} = C \setminus \{\theta\}$  and so  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$ . The result follows by Lemma 1.12(2).
  - (b)  $(1, 0) \notin C$ . We have that  $\{(2, 0), (1, 1)\} \subset C$ . As  $\text{Card}(C_n \setminus \mathfrak{A}) = n - 1 \geq 2$ , there exists  $\tau \in C_n \setminus \mathfrak{A}$  such that  $\tau_1, \tau_2 \geq 1$  or  $\tau = (n, 0)$ . If  $\tau_1, \tau_2 \geq 1$  let  $\theta = (1, 1)$ . If  $\tau = (n, 0)$  let  $\theta = (2, 0)$ . In any of the cases  $\theta \in C$  and  $\theta \cdot w \leq \tau \cdot w$ . We can take  $\mathcal{A} = C \setminus \{\theta\}$  and so  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$ . The result follows by Lemma 1.12(2).
- (3)  $\text{Card}(C) = M - j$  for  $3 \leq j \leq n - 1$ . We have two subcases:

- (a)  $C = B_1 \setminus \{(0, b_1), \dots, (0, b_{j-1})\}$ . Then  $\theta = (1, 0) \in C$  and  $\text{Card}(C_n \setminus \mathfrak{A}) = n+1-j \geq 2$ . Then there exists  $\tau \in C_n \setminus \mathfrak{A}$  such that  $\theta \cdot w \leq \tau \cdot w$ . We can take  $\mathcal{A} = C \setminus \{\theta\}$  and so  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$ . The result follows by Lemma 1.12(2).
- (b)  $C = B_1 \setminus \{(a_1, b_1), \dots, (a_{j-1}, b_{j-1})\}$  with at least one subindex  $i$ ,  $1 \leq i \leq j-1$  such that  $\theta = (a_i, b_i)$  satisfies that  $a_i \geq 1$ . In this case,  $\theta \in B_1 \setminus C$ . We can take  $\mathcal{A} = C$  and so  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$ . The result follows by Lemma 1.12(1).
- (4)  $\text{Card}(C) = M - (n+1)$ . In this situation there exists  $\theta \in B_1 \setminus C$  with  $\theta_1 \geq 1$ . In this case we can take  $\mathcal{A} = C$  and so  $\text{Card}(B_1 \setminus \mathcal{A}) \geq 2$ . The result follows by Lemma 1.12(1).  $\square$

**Corollary 1.16.** *Let  $f \in \mathbb{C}[x_1, x_2]$  be a weighted homogeneous polynomial of weight  $w \in \mathbb{N}^2$  and degree  $d$ . Suppose that  $d \geq 2w_1 \geq 2w_2 > 0$ . Let  $n \geq 3$  and  $g$  be a maximal minor of  $\text{Jac}_n(f)$ . Then  $g$  is a weighted homogeneous polynomial of degree greater or equal than  $d$ .*

*Proof.* By Theorem 1.7,  $g$  is a weighted homogeneous polynomial of degree  $dM - c$ . Propositions 1.14 and 1.15 imply  $d(M-1) \geq 2w_1(M-1) \geq c$ .  $\square$

## 2. DERIVATIONS OF THE HIGHER NASH BLOWUP LOCAL ALGEBRA

In this section we study the Lie algebra of derivations of the higher Nash blowup local algebra of a hypersurface defining an isolated singularity.

**Definition 2.1.** [HMYZ23, Definition 1.3]. Let  $f \in \mathbb{C}\{x_1, \dots, x_s\}$ . Denote

$$\mathcal{T}_n(f) = \mathbb{C}\{x_1, \dots, x_s\} / \langle f, \mathcal{J}_n(f) \rangle.$$

$\mathcal{T}_n(f)$  is called the higher Nash blowup local algebra of  $f$ . For  $n = 1$  this is also known as the Tjurina algebra of  $f$ . Moreover, denote as  $\mathcal{L}_n(f) = \text{Der}(\mathcal{T}_n(f))$ , i.e., the Lie algebra of derivations of  $\mathcal{T}_n(f)$ .

**Remark 2.2.** The higher Nash blowup local algebra is denoted as  $\mathcal{M}_n$  in [HMYZ23]. In this paper we change the notation in order to be consistent with the usual notation for the Tjurina algebra. In addition, it is also customary to use  $\mathcal{M}_1$  (or, more precisely,  $\mathcal{M}$ ) for the Milnor algebra.

Let  $f$  be a weighted homogeneous polynomial defining an isolated hypersurface singularity. Theorem 1.7 implies that  $\mathbb{C}[x_1, \dots, x_s] / \langle f, \mathcal{J}_n(f) \rangle$  is graded. There is an induced grading on  $\text{Der}(\mathbb{C}[x_1, \dots, x_s] / \langle f, \mathcal{J}_n(f) \rangle)$ , as we explain next.

**Remark 2.3.** Suppose that the hypersurface defined by  $f$  has an isolated singularity. Then  $\mathbb{C}[x_1, \dots, x_s] / \langle f, \mathcal{J}_n(f) \rangle$  is Artinian. This follows from the fact that the zero locus of  $\langle f, \mathcal{J}_n(f) \rangle$  coincides with the singular locus of the hypersurface [Dua17, Corollary 2.2].

**Lemma 2.4** ([XY96, Lemma 2.1]). *Let  $A = \bigoplus_{i=0}^t A_i$  be a graded commutative Artinian local algebra. Let  $L(A)$  be the derivation algebra of  $A$ . Then  $L(A)$  can be graded as follows:  $L(A) = \bigoplus_{k=-t}^t L_k$ , where  $L_k = \{D \in L(A) \mid D(A_i) \subset A_{i+k} \text{ for all } i\}$ .*

N. Hussain, G. Ma, S. S.-T. Yau, and H. Zuo proposed the following conjecture regarding the non-existence of negative weight derivations on  $\mathcal{L}_n(f)$ .

**Conjecture 2.5** ([HMYZ23, Conjecture 1.7]). *Let  $f \in \mathbb{C}[x_1, \dots, x_s]$  be a weighted homogeneous polynomial of weight  $w \in \mathbb{N}^s$  and degree  $d$  that defines an isolated hypersurface singularity. Suppose that  $d \geq 2w_1 \geq \dots \geq 2w_s > 0$ . Then there is no non-zero negative weight derivation on  $\mathcal{T}_n(f)$ , i.e.,  $\mathcal{L}_n(f)$  is non-negatively graded.*

The same authors verified Conjecture 2.5 in the case  $s = n = 2$  [HMYZ23, Theorem B]. The following theorem gives a positive answer to Conjecture 2.5 for  $s = 2$  and  $n \geq 3$ . Together, these results settle the conjecture for two variables.

**Theorem 2.6.** *Let  $f \in \mathbb{C}[x_1, x_2]$  be a weighted homogeneous polynomial of weight  $(w_1, w_2) \in \mathbb{N}^2$  and degree  $d$  that defines an isolated hypersurface singularity. Suppose that  $d \geq 2w_1 \geq 2w_2 > 0$ . Let  $n \geq 3$ . Then  $\text{Der}(\mathbb{C}[x_1, x_2]/\langle f, \mathcal{J}_n(f) \rangle)$  is non-negatively graded.*

*Proof.* Let  $\bar{D}$  be a derivation of negative degree  $-k$  of  $\mathbb{C}[x_1, x_2]/\langle f, \mathcal{J}_n(f) \rangle$ . Then  $\bar{D}$  corresponds to a homogeneous derivation  $D$  of  $\mathbb{C}[x_1, x_2]$  such that  $D(\langle f, \mathcal{J}_n(f) \rangle) \subset \langle f, \mathcal{J}_n(f) \rangle$  [YZ16a, Theorem 2.2]. Since  $D$  is homogeneous, there are weighted homogeneous polynomials  $h_1, h_2$  of degree  $d_i = -k + w_i$ ,  $i = 1, 2$ , such that  $D = h_1\partial_{x_1} + h_2\partial_{x_2}$ . It is known that  $D(\langle x_1, x_2 \rangle) \subset \langle x_1, x_2 \rangle$  [XY96, Lemma 2.5]. This fact, together with  $D$  being of negative degree and  $w_1 \geq w_2$ , implies that  $h_2 = 0$  and  $h_1 = cx_2^b$ , for some  $c \in \mathbb{C}$  and  $b \geq 1$ . Hence,  $D = cx_2^b\partial_{x_1}$ .

By the assumptions on the weight and degree of  $f$ , and the fact that  $f$  defines an isolated singularity, we have that  $f$  must be one of the following cases [CXY95, Lemma 2.1]:

- (1)  $f = x_1^m + a_1(x_2)x_1^{m-1} + \cdots + a_m(x_2)$ .
- (2)  $f = x_1^m x_2 + a_1(x_2)x_1^{m-1} + \cdots + a_m(x_2)$ .

By Corollary 1.16,  $f$  is a generator of smallest degree among the generators of  $\langle f, \mathcal{J}_n(f) \rangle$ . Since  $D(f) \in \langle f, \mathcal{J}_n(f) \rangle$  and  $D$  is of negative degree, it follows that  $D(f) = 0$ , i.e.,  $cx_2^b\partial_{x_1}(f) = 0$ . Using (1) and (2) we obtain  $\partial_{x_1}(f) \neq 0$ . We conclude that  $c = 0$  and so  $D = 0$ .  $\square$

**Remark 2.7.** Consider the notation of Theorem 2.6. If  $f \in \langle x_1, x_2 \rangle^3$ , then we can assume  $d \geq 2w_1 \geq 2w_2 > 0$  without loss of generality [YZ16b, Theorem 2.1], [Sai71].

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#### REFERENCES

- [Bar23] Paul Barajas. High-order derivations of the hasse-schmidt algebra. 2023. [arXiv: 2305.04537](https://arxiv.org/abs/2305.04537)
- [BD20] Paul Barajas and Daniel Duarte. On the module of differentials of order  $n$  of hypersurfaces. *J. Pure Appl. Algebra*, 224(2):536–550, 2020. DOI: [10.1016/j.jpaa.2019.05.020](https://doi.org/10.1016/j.jpaa.2019.05.020)
- [BJNB19] Holger Brenner, Jack Jeffries, and Luis Núñez-Betancourt. Quantifying singularities with differential operators. *Adv. Math.*, 358:106843, 89, 2019. DOI: [10.1016/j.aim.2019.106843](https://doi.org/10.1016/j.aim.2019.106843)
- [CMDGF21] Enrique Chávez-Martínez, Daniel Duarte, and Arturo Giles Flores. A higher-order tangent map and a conjecture on the higher Nash blowup of curves. *Math. Z.*, 297(3-4):1767–1791, 2021. DOI: [10.1007/s00209-020-02579-5](https://doi.org/10.1007/s00209-020-02579-5)
- [CCYZ20] Bingyi Chen, Hao Chen, Stephen S.-T. Yau, and Huaiqing Zuo. The nonexistence of negative weight derivations on positive dimensional isolated singularities: generalized Wahl conjecture. *J. Differential Geom.*, 115(2):195–224, 2020. DOI: [10.4310/jdg/1589853625](https://doi.org/10.4310/jdg/1589853625)
- [CHYZ20] Bingyi Chen, Naveed Hussain, Stephen S.-T. Yau, and Huaiqing Zuo. Variation of complex structures and variation of Lie algebras II: New Lie algebras arising from singularities. *J. Differential Geom.*, 115(3):437–473, 2020. DOI: [10.4310/jdg/1594260016](https://doi.org/10.4310/jdg/1594260016)
- [Che99] Hao Chen. Nonexistence of negative weight derivations on graded Artin algebras: a conjecture of Halperin. *J. Algebra*, 216(1):1–12, 1999. DOI: [10.1006/jabr.1998.7750](https://doi.org/10.1006/jabr.1998.7750)
- [CXY95] Hao Chen, Yi-Jing Xu, and Stephen S.-T. Yau. Nonexistence of negative weight derivation of moduli algebras of weighted homogeneous singularities. *J. Algebra*, 172(2):243–254, 1995. DOI: [10.1016/S0021-8693\(05\)80001-8](https://doi.org/10.1016/S0021-8693(05)80001-8)
- [CYZ19] Hao Chen, Stephen S.-T. Yau, and Huaiqing Zuo. Non-existence of negative weight derivations on positively graded Artinian algebras. *Trans. Amer. Math. Soc.*, 372(4):2493–2535, 2019. DOI: [10.1090/tran/7628](https://doi.org/10.1090/tran/7628)
- [dAD21] Hernán de Alba and Daniel Duarte. On the  $k$ -torsion of the module of differentials of order  $n$  of hypersurfaces. *J. Pure Appl. Algebra*, 225(8):Paper No. 106646, 7, 2021. DOI: [10.1016/j.jpaa.2020.106646](https://doi.org/10.1016/j.jpaa.2020.106646)
- [DGI20] Alexandru Dimca, Rodrigo Gondim, and Giovanna Ilardi. Higher order Jacobians, Hessians and Milnor algebras. *Collect. Math.*, 71(3):407–425, 2020. DOI: [10.1007/s13348-019-00266-1](https://doi.org/10.1007/s13348-019-00266-1)

- [DS15] Alexandru Dimca and Gabriel Sticlaru. Hessian ideals of a homogeneous polynomial and generalized Tjurina algebras. *Doc. Math.*, 20:689–705, 2015. DOI: [10.4171/dm/502](https://doi.org/10.4171/dm/502)
- [Dua17] Daniel Duarte. Computational aspects of the higher Nash blowup of hypersurfaces. *J. Algebra*, 477:211–230, 2017. DOI: [10.1016/j.jalgebra.2016.12.026](https://doi.org/10.1016/j.jalgebra.2016.12.026)
- [FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001. DOI: [10.1007/978-1-4613-0105-9](https://doi.org/10.1007/978-1-4613-0105-9)
- [GLS07] G.-M. Greuel, C. Lossen, and E. Shustin. *Introduction to singularities and deformations*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [GP17] Gert-Martin Greuel and Thuy Huong Pham. Mather-Yau theorem in positive characteristic. *J. Algebraic Geom.*, 26(2):347–355, 2017. DOI: [10.1090/jag/669](https://doi.org/10.1090/jag/669)
- [HMYZ23] Naveed Hussain, Guorui Ma, Stephen S.-T. Yau, and Huaiqing Zuo. Higher Nash blow-up local algebras of singularities and its derivation Lie algebras. *J. Algebra*, 618:165–194, 2023. DOI: [10.1016/j.jalgebra.2022.11.016](https://doi.org/10.1016/j.jalgebra.2022.11.016)
- [LDS24] Yann Le Dréau and Julien Sebag. Arc scheme and higher differential forms. *Osaka J. Math.*, 61(3):381–390, 2024.
- [LY24] Quy Thuong Lê and Takehiko Yasuda. Higher jacobian ideals, contact equivalence and motivic zeta functions. *Rev. Mat. Iberoam.*, 41(3): 1057–1080, 2025. DOI: [10.4171/rmi/1516](https://doi.org/10.4171/rmi/1516)
- [MYZ23] Guorui Ma, Stephen S.-T. Yau, and Huaiqing Zuo. Nonexistence of negative weight derivations of the local  $k$ -th Hessian algebras associated to isolated singularities. *Pacific J. Math.*, 323(1):129–172, 2023. DOI: [10.2140/pjm.2023.323.129](https://doi.org/10.2140/pjm.2023.323.129)
- [MY82] John N. Mather and Stephen S. T. Yau. Classification of isolated hypersurface singularities by their moduli algebras. *Invent. Math.*, 69(2):243–251, 1982. DOI: [10.1007/BF01399504](https://doi.org/10.1007/BF01399504)
- [Mei82] W. Meier. Rational universal fibrations and flag manifolds. *Math. Ann.*, 258(3):329–340, 1981/82. DOI: [10.1007/BF01450686](https://doi.org/10.1007/BF01450686)
- [Sai71] Kyoji Saito. Quasihomogene isolierte Singularitäten von Hyperflächen. *Invent. Math.*, 14:123–142, 1971. DOI: [10.1007/BF01405360](https://doi.org/10.1007/BF01405360)
- [SY90] Craig Seeley and Stephen S.-T. Yau. Variation of complex structures and variation of Lie algebras. *Invent. Math.*, 99(3):545–565, 1990. DOI: [10.1007/BF01234430](https://doi.org/10.1007/BF01234430)
- [XY96] Yi-Jing Xu and Stephen S.-T. Yau. Micro-local characterization of quasi-homogeneous singularities. *Amer. J. Math.*, 118(2):389–399, 1996. DOI: [10.1353/ajm.1996.0020](https://doi.org/10.1353/ajm.1996.0020)
- [Yas07] Takehiko Yasuda. Higher Nash blowups. *Compos. Math.*, 143(6):1493–1510, 2007. DOI: [10.1112/S0010437X0700276X](https://doi.org/10.1112/S0010437X0700276X)
- [Yau86] Stephen S.-T. Yau. Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities. *Math. Z.*, 191(4):489–506, 1986. DOI: [10.1007/BF01162338](https://doi.org/10.1007/BF01162338)
- [Yau91] Stephen S.-T. Yau. Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by  $\mathfrak{sl}(2, \mathbb{C})$  invariant polynomials. *Amer. J. Math.*, 113(5):773–778, 1991. DOI: [10.2307/2374785](https://doi.org/10.2307/2374785)
- [YZ16a] Stephen S.-T. Yau and Huai Qing Zuo. A sharp upper estimate conjecture for the Yau number of a weighted homogeneous isolated hypersurface singularity. *Pure Appl. Math. Q.*, 12(1):165–181, 2016. DOI: [10.4310/PAMQ.2016.v12.n1.a6](https://doi.org/10.4310/PAMQ.2016.v12.n1.a6)
- [YZ16b] Stephen S.-T. Yau and Huai Qing Zuo. Derivations of the moduli algebras of weighted homogeneous hypersurface singularities. *J. Algebra*, 457:18–25, 2016. DOI: [10.1016/j.jalgebra.2016.03.003](https://doi.org/10.1016/j.jalgebra.2016.03.003)

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