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MORSE FUNCTIONS CONSTRUCTED BY RANDOM WALKS

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Abstract. We construct random Morse functions on surfaces by random walk and compute related distributions. We study the space of Morse functions through these random variables. We consider subspaces characterized by the surfaces with boundary obtained by cutting the closed domain surface of the Morse function at the levels of regular values. We consider Morse functions having a bounded number of critical points and one single local minimum. We find a small set of Morse functions which are close enough to any other Morse function in the sense that they share the same characterizing surfaces with boundary.

1. Introduction

A Morse function on an n-dimensional closed manifold M is a smooth map $f\colon M \to \mathbb{R}$ which has the local form $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n \pm x_i^2$ around each singular point. Morse functions are important in studying differentiable manifolds, many constructions for them are related to topological properties of smooth manifolds. In the present paper, we deal with the space of Morse functions on closed orientable surfaces. We define some random variables of Morse functions indicating the number of boundary circle components and the genus of the surface with boundary obtained by cutting the surface along a level of a regular value (of the Morse function). By genus we mean the genus of the closed surface that we get by attaching disks to the boundary circle components. We apply an approach imparted by the probabilistic method [AS00], to obtain results about the subspaces of Morse functions defined from this perspective. We consider two Morse functions $f_{1,2}\colon M \to \mathbb{R}$ to be equivalent if there are some diffeomorphism $\varphi\colon M \to M$ and orientation preserving diffeomorphism $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $f_2 = \psi \circ f_1 \circ \varphi$. However, our process can yield many non-equivalent constructed Morse functions at the same time on a single surface (depending on the arrangement of critical values in $\mathbb{R}$), we explain this in detail later.

Random constructions for mathematical structures are often used when we want to show the existence of some structure with a special property, in combinatorics this method was initiated by T. Szele and P. Erdős in the '40s. For example, random subgraphs can be used to prove many statements [AS00] in graph theory, random 3-manifolds were studied in [DT06a, DT06b], random groups in [Gr00, Gr03] and random knots in [EHLN16].

If we have a simple construction for all possible studied mathematical objects (for example special diagrams for knots or Heegaard splitting for 3-manifolds), then we can define some probabilities for taking basic pieces one-by-one and building the objects from them. This corresponds to some random walk since we make possible choices among elements with given probabilities until some condition is satisfied and then finally we have the complete object constructed randomly. In this way from obtaining probabilities of creating random objects, we get a probability space and we can work with random variables and their expected values, for example. The simplest idea to apply these is that if the expected value of a random variable is greater (resp. smaller) than or equal to some number, then the same holds for the value of the random variable

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at some object as well. So we can prove the existence of some special object whose properties are expressed by some random variable.

In this paper, we construct random Morse functions on surfaces by using random walk and compute some related probabilities and distributions. We consider Morse functions with one single local minimum. During the random walk process, we can step in the set $[0,\infty)^2 \cap \mathbb{Z}^2$ in three directions: right, left and up-and-left with probabilities $p_r, p_l, p_d > 0$, respectively, with $p_r + p_l + p_d = 1$ all along starting at $(1,0)$. Each step corresponds to attaching prescribed pieces of Morse functions with one critical point (an upside-down pair of pants, a local maximum and a usual pair of pants, respectively) to get finally Morse functions on a closed surface when the random walk reaches some $(0,z)$ for the first time. At the attaching of the Morse function pieces we do not specify exactly which boundary circle we use so in this sense our model does not describe completely the construction of Morse functions as we imagine usually when we sketch one. In other words, the constructed Morse function for one specified walk is not unique. So in fact we construct sets of Morse functions having the order (according to the orientation of the target $\mathbb{R}$) of specific indices of critical values in common. We do not study the number of Morse functions in these sets in the present paper.

The random walk would be a random variable of random Morse functions and we compute properties of this random variable. If $\mathcal{O}$ is the σ-algebra of walks on the set Ω' of walks and on the set Ω of Morse functions with one local minimum we have the map $W\colon \Omega \to \mathcal{P}(\Omega')$ assigning to a Morse function the set of corresponding infinite walks (reaching the appropriate $(0,z)$ for the first time) by the above indicated procedure, then the σ-algebra $\{W^{-1}(A) : A \in \mathcal{O}\}$ yields a probability space on Morse functions with the measure $P(W^{-1}(A)) = P(A)$ if $P(W(\Omega)) = 1$. Note that the condition $p_l + p_d \geq p_r$ implies that the walk will arrive back to some $(0,z)$ with probability 1. So the map $W\colon \Omega \to \mathcal{P}(\Omega')$ is almost like a random variable (that is a measurable map) of random Morse functions. A random variable of walks is a variable of Morse functions too (if the variable depends only on the walk until reaching $(0,z)$ for the first time), we compute some distributions and expected values of them in Proposition 3.1.

For example, in Proposition 3.1 we obtain that the expected number of the critical points of a random Morse function is $1 + \frac{1}{p_l + p_d - p_r}$ if $p_l + p_d > p_r$. In Proposition 3.3 we prove that if $p_l + p_d > p_r$, then the expected genus of the domain surface of a random Morse function with one local minimum is equal to

$$\frac{p_d + (p_l - p_d)(p_l + p_d - p_r)}{2(p_l + p_d)(p_l + p_d - p_r)}.$$

To get large enough values by these formulas we have to choose the probabilities p_r, p_l, p_d so that $p_l + p_d - p_r$ is small enough. For example with $p_d = 1/2, p_l = 1/20, p_r = 9/20$ we have that an average Morse function with one local minimum has 11 critical points and the genus of its domain surface is around 4.136. For example, the cobordism class in $\mathbb{Z}$ of a Morse function, see [IS03], is then expectedly equal to approximately 0.364 not only by Corollary 3.2 but also because by an easy summation we have that the number of critical points minus two is always equal to twice the sum of the cobordism class and the genus (if the Morse function has only one local minimum).

We would like to find an optimal set $\widetilde{\mathcal{D}}$ of Morse functions on a given oriented surface of genus $g \geq 0$ having the property that every Morse function is “close” to $\widetilde{\mathcal{D}}$ in a sense. We will do this on the level of random walks instead of Morse functions and we are looking for such an optimal set of walks $\mathcal{D}$. We define a graph of random walks $\mathcal{G}$ such that the i-th vertex corresponds to a walk w_i and so corresponds to many Morse functions each of which is mapped to the walk w_i by the random variable W. The optimal set of walks $\mathcal{D}$ is nothing else just a dominating set (a set of vertices being connected to all of the vertices) of the graph $\mathcal{G}$. The optimal set $\widetilde{\mathcal{D}}$ can be obtained then from the W-preimage of $\mathcal{D}$. The graph $\mathcal{G}$ is like a string graph of the walks or the vertex intersection graph of the walks for Morse functions (vertex intersection graphs were introduced by [ACGLLS12]). We obtain in Corollary 3.6 that in the set of Morse functions with at most $N \geq 2g+2$ critical points and one local minimum on a genus $g \geq 2$ surface there is such a $\widetilde{\mathcal{D}}$ consisting of at most

$$M_{N-2}\left(1 - \frac{\Delta}{(\Delta+1)^{1+\frac{1}{\Delta}}}\right)$$

Morse functions, where $\Delta + 1$ is equal to the g-th Catalan number and M_{N-2} is equal to $\sum_{k=g}^{(N-2)/2} \frac{1}{2k+1}\binom{2k+1}{k+1}\binom{k}{g}$. This follows from our Theorem 3.5, where we apply [AS00, Chapter I, Theorem 1.2.2].

The "closeness" of a Morse function to the set $\widetilde{\mathcal{D}}$ means that cutting an arbitrary Morse function $f \notin \widetilde{\mathcal{D}}$ at the level set $f^{-1}(a)$ of some appropriate regular value $a \in \mathbb{R}$ the obtained surface $f^{-1}((-\infty, a])$ has

(1) genus G, that is if we close the surface by disks it has genus G, and
(2) C copies of boundary circles

such that the pair (C, G) can be obtained by similarly cutting some Morse function contained in $\widetilde{\mathcal{D}}$. The cases of $1 \leq G \leq g-1$ are more interesting for us than the cases of $G = 0$ and $G = g$.

The paper is organized as follows. In Section 2 we give our basic definitions and in Section 3 we prove our main results.

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2. Preliminaries

On the plane $\mathbb{R}^2$ let us consider the set

$$\mathcal{S} = \{(x, y) : x, y \in \mathbb{Z} \text{ and } x \geq 1, y \geq 0\},$$

where we start a random walk at the point $S_0 = (1, 0)$, we can step to the right, to the left and to up-and-left with positive probabilities p_r, p_l and p_d, respectively, where $p_r + p_l + p_d = 1$. We imagine that during these steps a surface, and its corresponding height functions (Morse functions), are built in the following way.

Algorithm. We stay in the set $\mathcal{S}$ all along at the choices of random steps. The n-th step is denoted by S_n. In each step, we modify possibly many Morse functions simultaneously. If at some step $S_n = (1, y)$, $n \geq 0$, we step outside of $\mathcal{S}$ (which can only happen at a left or diagonal step) for the first time, then we consider our Morse functions to be constructed by attaching one local maximum to each of them and we finish the process. Then we say that the random walk had n steps. Then the constructed Morse functions will have $n+2$ critical points. The Morse functions are built as follows. At the starting point we have one local minimum mapped to $\mathbb{R}$ as usual and at each step we raise our Morse functions:

(1) at a step to the right we raise our Morse functions by one pair of pants (the pants are upside down) attached to a circle boundary component,
(2) at a step to the left we attach one local maximum to some circle boundary component and
(3) at a step to up-and-left we attach a pair of pants (in the position as pants usually are) to two circle boundary components.

An example can be seen in Figure 1.

It is easy to see that Morse functions on a connected closed orientable surface are constructed when the random walk arrives to a point of the form $(1, y)$ after n steps, where $n \geq 0$, and we add one local maximum to the single boundary circle of the surface. This surface is unique by Morse homology and the classification of closed orientable surfaces. By simply counting the boundary components after each step, we see that a left or diagonal move reduces the boundary components by 1 and a right move increases them by 1, so $S_n = (x, y)$ exactly if we have x circle boundaries and the surface with boundary has genus y (by a simple argument using Morse homology). Of course, in this model the exact locus of the attaching at each step is undefined if there are more boundary components than attaching circles.

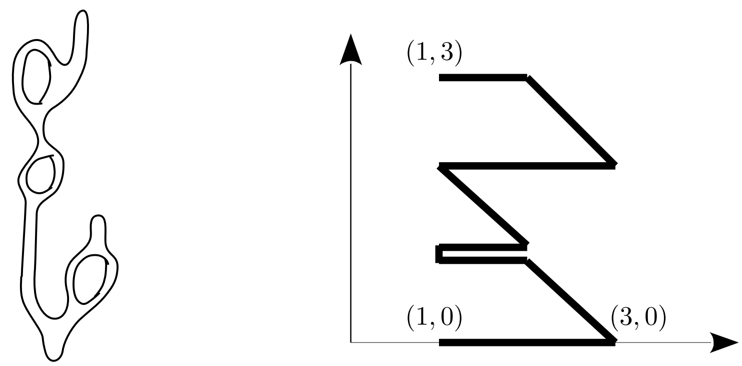

FIGURE 1. A Morse function as a height function on the left and the walk corresponding to it on the right. The length of the walk is equal to 10, the number of critical points is equal to 12. The genus of the domain surface is equal to 3. The walk starts at $(1, 0)$ and stops at $(1, 3)$. The two bottom indefinite critical points correspond to the first two steps to the right starting at $(1, 0)$, the third indefinite critical point (that is the first pair of pants in usual position resulting a twice punctured torus) corresponds to the step $(3, 0) \to (2, 1)$, etc.

Obviously the number of critical points of the Morse function under construction after the n-th step in $\mathcal{S}$ is equal to $n+1$, where $n \geq 0$, and finally we attach a local maximum when we step out from $\mathcal{S}$. So at the end of the process the Morse function on the closed surface has $n+2$ critical points if we had $S_n = (1, y)$ and then we left $\mathcal{S}$. If for some $n \geq 0$ we have $S_n = (x, y)$, then the number of index 1 critical points with usually positioned (not upside down) pair of pants is equal to y. Also then the number of right steps minus the number of left steps minus the number of up-and-left steps is equal to $x - 1$.

Lemma 2.1. *If $S_n = (x, y)$, then*
(a) *the number r of right steps is equal to $(n + x - 1)/2$,*
(b) *the number l of left steps is equal to $(n - x + 1)/2 - y$,*
(c) *the number d of up-left steps is equal to y.*
So $S_n = (1, y)$ after $n \geq 0$ steps if and only if $r = n/2$, $l = n/2 - y$ and $d = y$.

Proof. Of course $r + l + d = n$, $r - l - d = x - 1$ and $d = y$ give the result. □

We want to find an optimal set $\widetilde{\mathcal{D}}$ of Morse functions on a given closed oriented surface of genus $g \geq 0$ having the property that any Morse function is close to $\widetilde{\mathcal{D}}$ in some sense. We are going to do this on the level of random walks instead of Morse functions and we are looking for such an optimal set of walks $\mathcal{D}$. Recall from Introduction that we have the measurable map $W\colon \Omega \to \Omega'$ assigning to a Morse function the corresponding walks by the above indicated procedure. We define a graph of random walks $\mathcal{G}$ such that every vertex corresponds to a walk w_i and so corresponds to many Morse functions each of which is mapped to the walk w_i by the random variable W. The optimal set $\widetilde{\mathcal{D}}$ would be then the W-preimage of our optimal set of walks $\mathcal{D}$. We also put some restrictions on the genus g and the maximal number N of critical points of Morse functions as follows. We define this optimal set of Morse functions with the help of constructing a graph.

Definition 2.2. For $N \geq 2$ and $g \geq 0$ let $\widetilde{\mathcal{G}}_{N,g}$ be the graph

(1) whose vertices are the Morse functions with at most N critical points on a closed surface of genus g and
(2) whose edges are exactly between those vertices which are Morse functions f_1 and f_2 with the property that at the level sets $f_1^{-1}(a_1)$ and $f_2^{-1}(a_2)$ of some appropriate regular values $a_1 \in \mathbb{R}$ for f_1 and $a_2 \in \mathbb{R}$ for f_2 the obtained surfaces $f^{-1}((-\infty, a_1])$ and $f^{-1}((-\infty, a_2])$ have the same number of boundary circles and the same genus. This common genus G is required to satisfy $1 \leq G \leq g-1$.

A dominating set in $\widetilde{\mathcal{G}}_{N,g}$, that is a subset $\widetilde{\mathcal{D}}_{N,g}$ of the vertices of $\widetilde{\mathcal{G}}_{N,g}$ such that any vertex of $\widetilde{\mathcal{G}}_{N,g}$ is in $\widetilde{\mathcal{D}}_{N,g}$ or has a neighbor in $\widetilde{\mathcal{D}}_{N,g}$, is called a *dominating set* of Morse functions.

Remark 2.3. It is possible to describe the dominating sets of Morse functions more directly. Take a set $\widetilde{\mathcal{D}}_{N,g}$ of Morse functions satisfying the condition that cutting an arbitrary Morse function $f \notin \widetilde{\mathcal{D}}_{N,g}$ at the level set $f^{-1}(a)$ of some appropriate regular value $a \in \mathbb{R}$ the obtained surface $f^{-1}((-\infty, a])$ has

(1) genus G, that is if we close the surface by disks it has genus G, and
(2) C copies of boundary circles

such that the pair (C, G) can be obtained by similarly cutting some Morse function contained in $\widetilde{\mathcal{D}}_{N,g}$. We require $1 \leq G \leq g-1$. For all Morse functions we put the restriction that the number of their critical points has to be at most $N \geq 2$. Such a set $\widetilde{\mathcal{D}}_{N,g}$ of Morse functions on a genus $g \geq 0$ domain surface is a dominating set of Morse functions.

A dominating set in $\widetilde{\mathcal{G}}_{N,g}$ corresponds clearly to a dominating set $\widetilde{\mathcal{D}}_{N,g}$ of Morse functions.

We require $1 \leq G \leq g-1$ because the local minimum or the index one critical points for the lowest function values can be easily common at two different Morse functions, similarly near the local maxima we do not see too much interesting differences between two Morse functions on the same genus g surface.

Remark 2.4. Note that if $f \in \widetilde{\mathcal{D}}_{N,g}$ where $\widetilde{\mathcal{D}}_{N,g}$ is dominating, then the set

$$\widetilde{\mathcal{D}}_{N,g} - \{g \in W^{-1}(W(f)) : g \neq f\}$$

is also dominating with less Morse functions.

We define the notions analogous to the graph $\widetilde{\mathcal{G}}_{N,g}$ and the dominating sets $\widetilde{\mathcal{D}}_{N,g}$ for the random walks too. We define the graph $\mathcal{G}_{N-2,g}$ as follows.

Definition 2.5. Let $N \geq 2$ and $g \geq 0$. The vertices of the graph $\mathcal{G}_{N-2,g}$ are in bijection with the random walks of length at most $N-2$ in $\mathcal{S}$ starting from $(1,0)$ and finishing at the final point $(1,g)$. The edges of $\mathcal{G}_{N-2,g}$ are exactly between those vertices which correspond to walks w_1 and w_2 such that w_1 and w_2 intersect each other in a point $(C,G) \in \mathcal{S}$, where $1 \leq G \leq g-1$.

In fact, we are looking for dominating sets $\mathcal{D}$ in the graph $\mathcal{G}_{N-2,g}$. Then $W^{-1}(\mathcal{D})$ is a dominating set $\widetilde{\mathcal{D}}_{N,g}$ in $\widetilde{\mathcal{G}}_{N,g}$. The graph $\mathcal{G}_{N-2,g}$ is almost like a string graph of the walks and also the vertex intersection graph of the walks for Morse functions in $\mathcal{S}$. For string or vertex intersection graphs of paths, see, for example [ACGLLS12].

3. Results

Observe that if $S_n = (1, y)$, then obviously n is even and also $0 \le y \le n/2$. In this paper, we consider Morse functions with only one local minimum.

Proposition 3.1. *Suppose that* $p_l + p_d \ge p_r$.

(1) *The probability that the random walk starting at* $(1,0)$ *arrives to a point of the form* $(0,z)$ *is equal to* 1.

(2) *Let* $n \ge 0$ *be an even integer. The probability that from a walk a random Morse function is created with* $n+2$ *critical points on a closed surface is equal to*

$$\frac{1}{n+1}\binom{n+1}{(n+2)/2} p_r^{n/2}(p_l+p_d)^{(n+2)/2}.$$

(3) *The expected number of the critical points of a random Morse function is*

$$1 + \frac{1}{p_l + p_d - p_r}$$

if $p_l + p_d > p_r$ *and* ∞ *if* $p_l + p_d = p_r$.

(4) *Let* $y \ge 0$ *and* $n \ge 0$ *be an even integer. The probability that during the walk a random Morse function is created with* $n+2$ *critical points on a closed surface of genus* y *(so* n *is necessarily even and the walk finishes at* $S_{n+1} = (0,z)$, $S_n = (1,y)$ *and we start at* $(1,0)$*) is equal to*

$$\frac{1}{n+1}\binom{n+1}{(n+2)/2} p_r^{n/2}(p_l+p_d)\binom{n/2}{y} p_l^{n/2-y} p_d^{y}.$$

(5) *The expected number of local maxima of a random Morse function with one local minimum is equal to*

$$\frac{p_d(p_l+p_d-p_r) + p_l/2}{(p_l+p_d)(p_l+p_d-p_r)}$$

if $p_l + p_d > p_r$.

Proof. For (1) note that it is well-known that if $p_l + p_d \ge p_r$, then the probability of hitting $(0,z)$ for some z is equal to 1, since we can project our points in $\mathbb{Z}\times\mathbb{Z}$ to the first coordinates to get a 1-dimensional random walk and we can apply the well-known theory, see [Fe68, page 347 (2.8)].

For (2) the distribution of the first passage at 0 of a 1-dimensional random walk with initial position $z > 0$ is well-known [Fe68, Chapter XIV, Section 4] and if it happens at the $(n+1)$-th step with $z = 1$, then it is equal to

$$\frac{1}{n+1}\binom{n+1}{(n+2)/2} p_r^{n/2} q^{(n+2)/2},$$

where $q = p_l + p_d$ since we project to the first coordinates our position in $\mathbb{Z}\times\mathbb{Z}$.

For (3) we can use [Fe68, Chapter XIV, Section 3]. If our walking process has length n in the sense of our algorithm on page 3, then the resulting Morse function has $n+2$ critical points while the length of the random walk is equal to $n+1$. So by [Fe68, Chapter XIV, Section 3] since the expected duration of the walking is $\frac{1}{p_l+p_d-p_r}$, the expected number of critical points is equal to $1 + \frac{1}{p_l+p_d-p_r}$ if $p_l + p_d > p_r$. If $p_l + p_d = p_r$, then the expected duration is infinite.

For (4) note that the number of walks in $\mathcal{S}$ is equal to

$$\frac{1}{n+1}\binom{n+1}{(n+2)/2}2^{n/2}$$

since the number of walks in 1-dimension (after the projection to the first coordinates our position in $\mathbb{Z}\times\mathbb{Z}$) is equal to $\frac{1}{n+1}\binom{n+1}{(n+2)/2}$ by basic combinatorics. Consider (2) and that in order to reach height y, that is genus y, we can have y number of up-and-left steps in $\binom{n/2}{y}$ different ways so the number of walks is equal to

$$\frac{1}{n+1}\binom{n+1}{(n+2)/2}\binom{n/2}{y}.$$

Each walk has probability $p_r^{n/2}(p_l+p_d)p_l^{n/2-y}p_d^y$ so we get the result.

For (5) the expected value is equal to

$$\sum_{n\geq 0,\, 2|n}\frac{1}{n+1}\binom{n+1}{(n+2)/2}p_r^{n/2}(p_l+p_d)\sum_{0\leq y\leq n/2}\left(\frac{n}{2}-y+1\right)\binom{n/2}{y}p_l^{\frac{n}{2}-y}p_d^y$$

by the standard way to compute the expected value since for a constructed Morse function the number of local maxima is equal to $\frac{n}{2}-y+1$ by Lemma 2.1 considering genus y surfaces. We have

$$\sum_{0\leq y\leq n/2}\left(\frac{n}{2}-y+1\right)\binom{n/2}{y}p_l^{\frac{n}{2}-y}p_d^y=\left(\frac{\partial}{\partial x}\sum_{0\leq y\leq \frac{n}{2}}\binom{n/2}{y}x^{\frac{n}{2}-y+1}p_d^y\right)_{x=p_l}=$$
$$\frac{\partial}{\partial x}x(x+p_d)^{n/2}|_{x=p_l}=(p_l+p_d)^{\frac{n}{2}}+p_l\frac{n}{2}(p_l+p_d)^{\frac{n}{2}-1}=(p_d+\frac{n+2}{2}p_l)(p_l+p_d)^{\frac{n}{2}-1}$$

if $n/2\geq 1$, otherwise for $n=0$ we have 1. So the expected number of the local maxima is equal to

$$p_l+p_d+\sum_{n\geq 1,\, 2|n}\frac{1}{n+1}\binom{n+1}{(n+2)/2}p_r^{n/2}(p_l+p_d)(p_d+\frac{n+2}{2}p_l)(p_l+p_d)^{\frac{n}{2}-1}=$$
$$\sum_{n\geq 0,\, 2|n}^{\infty}\frac{1}{n+1}\binom{n+1}{(n+2)/2}p_r^{\frac{n}{2}}(p_l+p_d)^{\frac{n}{2}}(p_d+\frac{n+2}{2}p_l)=$$
$$p_d\sum_{n\geq 0,\, 2|n}^{\infty}\frac{1}{n+1}\binom{n+1}{(n+2)/2}p_r^{\frac{n}{2}}(p_l+p_d)^{\frac{n}{2}}+\frac{p_l}{2}\sum_{n\geq 0,\, 2|n}^{\infty}\frac{n+2}{n+1}\binom{n+1}{(n+2)/2}p_r^{\frac{n}{2}}(p_l+p_d)^{\frac{n}{2}}=$$
$$\frac{p_d}{p_l+p_d}+\frac{p_l}{2}\frac{1}{(p_l+p_d)(p_l+p_d-p_r)}=\frac{p_d(p_l+p_d-p_r)+p_l/2}{(p_l+p_d)(p_l+p_d-p_r)}.$$

□

Recall that the cobordism group of Morse functions on oriented surfaces is isomorphic to $\mathbb{Z}$ and the cobordism class of a Morse function is given by the number of local maxima minus the number of local minima [IS03].

Corollary 3.2.

(1) The expected cobordism class in $\mathbb{Z}$ of a random Morse function with one local minimum is equal to

$$\frac{p_d(p_l+p_d-p_r)+p_l/2}{(p_l+p_d)(p_l+p_d-p_r)}-1=\frac{\frac{p_l}{2}-p_l(p_l+p_d-p_r)}{(p_l+p_d)(p_l+p_d-p_r)}.$$

(2) If $p_l + p_d > p_r$, then the expected number of index one critical points of a random Morse function with one local minimum is equal to

$$1 + \frac{1}{p_l + p_d - p_r} - \frac{p_d(p_l + p_d - p_r) + p_l/2}{(p_l + p_d)(p_l + p_d - p_r)} - 1 = \frac{\frac{p_l}{2} + p_d(1 - p_l - p_d + p_r)}{(p_l + p_d)(p_l + p_d - p_r)}$$

because we get the number of index one critical points by subtracting from the number of all critical points the number of index 0 or 2 critical points.

Proposition 3.3. *If $p_l + p_d > p_r$, then the expected genus of the domain surface of a random Morse function with one local minimum is equal to*

$$\frac{p_d + (p_l - p_d)(p_l + p_d - p_r)}{2(p_l + p_d)(p_l + p_d - p_r)}.$$

Proof. By the fact that the cobordism class of a Morse function is equal to the number of index one critical points minus twice the number of index one critical points in the usual pair of pants position, see [IS03], and that the genus is equal to the number of index one critical points in the usual pair of pants position because we have always one local minimum, we get

$$\frac{1}{2}\left(\frac{\frac{p_l}{2} + p_d(1 - p_l - p_d + p_r)}{(p_l + p_d)(p_l + p_d - p_r)} - \frac{\frac{p_l}{2} - p_l(p_l + p_d - p_r)}{(p_l + p_d)(p_l + p_d - p_r)}\right) =$$
$$\frac{p_d(1 - p_l - p_d + p_r) + p_l(p_l + p_d - p_r)}{2(p_l + p_d)(p_l + p_d - p_r)} = \frac{p_d + (p_l - p_d)(p_l + p_d - p_r)}{2(p_l + p_d)(p_l + p_d - p_r)}.$$

□

Let $g \geq 0$ and $2g + 2 \leq N$, where recall that N is the upper bound for the number of critical points. By the previous statements the number of walks of length at most $N - 2$ is equal to

$$\sum_{\substack{2g \leq n \leq N-2 \\ 2|n}} \frac{1}{n+1}\binom{n+1}{\frac{n}{2}+1}\binom{n/2}{g}$$

and denote this number by M_{N-2}. Let Δ denote the g-th Catalan number minus one, that is

$$\Delta + 1 = \frac{1}{2g+1}\binom{2g+1}{g+1} = \frac{(2g)!}{(g+1)!g!}.$$

Then of course we have

$$\Delta = M_{2g} - 1.$$

Remark 3.4. Let $g \geq 2$ and $2g \leq N - 2$. Then in the graph $\mathcal{G}_{N-2,g}$ the minimum degree is at least one because of the following. If the first step during the walk from height 0 to height 1 happens at $(2,0)$, then the next step is from $(1,1)$ to $(2,1)$ and we can change these last two steps to $(2,0) \to (3,0) \to (2,1)$ to get a different walk with the same length intersecting each other at height $(2,1)$. If the first step during the walk from height 0 to height 1 happens at some $(a,0) \to (a-1,1)$ with $a \geq 3$, then the previous step is $(a-1,0) \to (a,0)$ or $(a+1,0) \to (a,0)$ and we can change the steps $(a-1,0) \to (a,0) \to (a-1,1)$ or $(a+1,0) \to (a,0) \to (a-1,1)$, respectively, to $(a-1,0) \to (a-2,1) \to (a-1,1)$ or $(a+1,0) \to (a,1) \to (a-1,1)$, respectively. Either way we get another walk with the same length intersecting the first walk at $(a-1,1)$. So in $\mathcal{G}_{N-2,g}$ the minimum degree is at least one.

Theorem 3.5. *Let $g \geq 2$ and $2g \leq N - 2$. Then there exists a dominating set $\mathcal{D}$ of the graph $\mathcal{G}_{N-2,g}$ of at most*

$$M_{N-2}\left(1 - \frac{\Delta}{(\Delta+1)^{1+\frac{1}{\Delta}}}\right)$$

vertices.

Proof. By [AS00, Chapter I, Theorem 1.2.2] and the explanation after it, in a graph of n vertices and minimum degree at least $\delta \geq 1$ there is a dominating set of at most

$$n\left(1-(\delta+1)^{-1/\delta}+(\delta+1)^{-(\delta+1)/\delta}\right)=n\left(1-\frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}}\right)$$

vertices as we can see easily. To show the statement we have to compute the number of vertices n and we have to show that the minimum degree in $\mathcal{G}_{N-2,g}$ is actually at least Δ. Note that $\Delta \geq 1$ because

$$\Delta+1=\frac{(2g)!}{(g+1)!g!}=\frac{2g(2g-1)\cdots(g+2)}{g!}\geq\frac{(g+2)(g+1)\cdots 4}{g!}=(g+2)(g+1)/6$$

for $g \geq 2$ so $\Delta+1 \geq (g+2)(g+1)/6 \geq 2$. As we mentioned, by the proof of Proposition 3.1 (4) the number of vertices n is equal to M_{N-2}. Also the minimum degree of $\mathcal{G}_{N-2,g}$ is at least Δ because every walk intersects another Δ since if the last step of a walk from height $g-1$ to height g is at $(a+1,g-1)\to(a,g)$, where $a \geq 1$, then all the possible simplest walks from $(a,0)$ to (a,g) and an additional walk from $(1,0)$ to $(a,0)$ give at least $\Delta+1$ walks (and our original walk is maybe among them). By “simplest” we mean the walks that arise when the walking length is equal to $2g$, see Figure 2. □

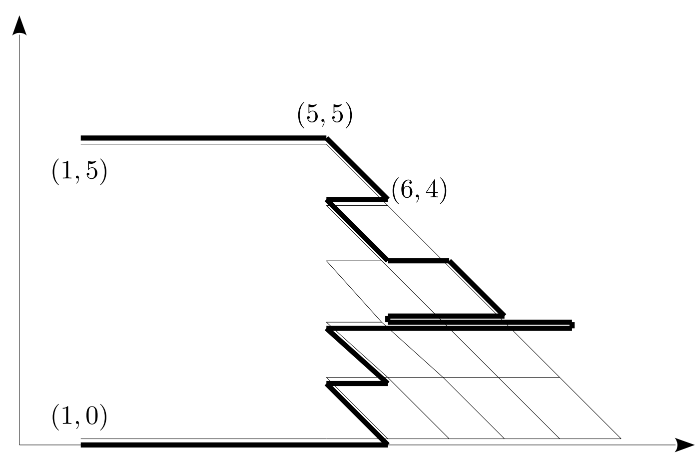

FIGURE 2. A walk from $(1,0)$ to $(1,5)$ as a bold zigzag. It reaches height 5 at the step $(6,4)\to(5,5)$. Then all the indicated zigzag walks (using only right and up-and-left steps) are such that they go from $(5,0)\to(5,5)$ and have shortest length 10. Their number is equal to $\Delta+1$. All of them intersect the bold walk at $(6,4)$ as well.

Corollary 3.6. There is a dominating set $\widetilde{\mathcal{D}}_{N,g}$ of the Morse functions with at most $N \geq 2g+2$ critical points and one local minimum on a genus $g \geq 2$ surface consisting of at most

$$M_{N-2}\left(1-\frac{\Delta}{(\Delta+1)^{1+\frac{1}{\Delta}}}\right)$$

Morse functions. This is true since $W^{-1}(\mathcal{D})$ is such a set if we take only one Morse function in each $W^{-1}(w)$, where $w \in \mathcal{D}$, see Remark 2.4.

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