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NON-INVERTIBLE QUASIHOMOGENEOUS SINGULARITIES AND THEIR LANDAU-GINZBURG ORBIFOLDS

ANTON RAROVSKII

Abstract. Based on the classification of quasihomogeneous singularities, any polynomial f defining such a singularity can be decomposed as $f = f_\kappa + f_{add}$. The polynomial f_κ takes a specific form, whereas f_{add} is constrained only by the requirement that the singularity of f should be isolated. The polynomial f_{add} is zero if and only if f is invertible; otherwise, in the non-invertible case, f_{add} may be arbitrarily complicated. This paper investigates all possible polynomials f_{add} for a given non-invertible f. For a fixed f_κ, we introduce a specific, small collection of monomials that constitute f_{add}, which guarantees that the polynomial $f = f_\kappa + f_{add}$ defines an isolated quasihomogeneous singularity. Furthermore, if $(f, \mathbb{Z}/2\mathbb{Z})$ is a Landau-Ginzburg orbifold with such a non-invertible polynomial f, we provide a quasihomogeneous polynomial $\bar{f}$ that satisfies the orbifold equivalence $(f, \mathbb{Z}/2\mathbb{Z}) \sim (\bar{f}, \{\mathrm{id}\})$. We also present an explicit isomorphism between the corresponding Frobenius algebras.

1. Introduction

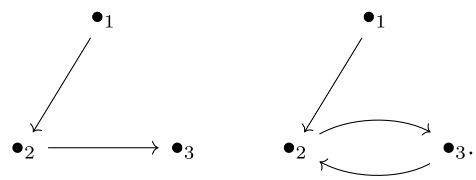
Let f be a quasihomogeneous polynomial defining an isolated singularity at the origin (in the remainder of this article, we call such f *non-degenerate*) and let G be the group of its symmetries. Then the pair (f, G) is called a *Landau–Ginzburg orbifold*. In the early 1990s, physicists began investigating such objects (cf. [IV, V, Witt]) and their applications in mathematical physics. Subsequently, interest in Landau–Ginzburg orbifolds spread among mathematicians. Interestingly, choosing f as a representative of a special class of polynomials, called *invertible*, leads to many well-known results in singularity theory, algebraic geometry, and mirror symmetry for Landau–Ginzburg orbifolds associated with f (cf. [BT1, BTW2, EGZ, BI, FJJS, KPA]). For instance, for diagonal groups G, there exists a well-defined dual pair $(\tilde{f}, \tilde{G})$ which is called the *BHK-dual* (see [BHu, BHe]), and a mirror isomorphism between their Frobenius algebras (see [Kr]).

Non-invertible polynomials. In this paper, we focus on a different class of polynomials: those that are non-degenerate yet non-invertible. As of today, there are relatively few results in singularity theory and mirror symmetry concerning these polynomials f and their corresponding Landau–Ginzburg orbifolds (f, G) (cf. [BT2, ET2]). According to [HK], any non-degenerate and non-invertible polynomial can be written as $f = f_\kappa + f_{add}$, where f_κ is constructed from a graph of the form $\Gamma_f = \sqcup \Gamma_{f_i}$, with each Γ_{f_i} being a «*Loop with Branches*» (see Figure 1), and $f_{add} = f - f_\kappa$.

As an example, let us consider the two polynomials

$$f_1 = x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} \qquad \text{and} \qquad f_2 = x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_2 + \varepsilon x_1^{b_1} x_3^{b_3}.$$

For f_2 to be non-degenerate, we require $\varepsilon \in \mathbb{C}^*$ and $b_1 q_1 + b_3 q_3 = 1$, where (q_1, q_2, q_3) is the set of weights of f (see Remark 2). In this case, f_1 is an example of an invertible polynomial, while f_2 is an example of a non-invertible polynomial. The corresponding graphs are



More specifically, $(f_1)_\kappa = f_1$, $(f_1)_{add} = 0$, $(f_2)_\kappa = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_2$, and $(f_2)_{add} = \varepsilon x_1^{b_1}x_3^{b_3}$.

Suppose now that we start from a quasihomogeneous f_κ, which corresponds to a graph of a quasihomogeneous singularity. If f_κ is non-invertible, then it must be degenerate (i.e., it defines a non-isolated singularity at the origin). In this paper, we introduce Theorem 19, which provides an admissible collection of monomials in f_{add} that renders $f = f_\kappa + f_{add}$ non-degenerate. Note that adding f_{add} given by Theorem 19 is not the only method to construct a non-degenerate polynomial from f_κ. However, our method is advantageous because it allows us to control which monomials the polynomial f_{add} consists of and on which variables it depends. Moreover, we introduce Theorem 22, which proposes a method to construct f_{add} involving a relatively small number of monomials. In this scenario, f_{add} would depend solely on variables whose indices follow a very specific logic: an arrow from the vertex that corresponds to the indexed variable must end at a vertex with index on the loop.

Orbifold equivalence. In the second part of the paper, we apply the aforementioned constructions to study the corresponding Landau–Ginzburg orbifolds. Specifically, that section will be devoted to the orbifold equivalence $(f, G) \sim (\bar{f}, \{\mathrm{id}\})$ between Landau–Ginzburg orbifolds, which could potentially be useful in studying Landau–Ginzburg orbifolds with non-trivial groups G. Orbifold equivalence could roughly be understood as an equivalence $\mathrm{MF}(\bar{f}) \cong \mathrm{MF}_G(f)$ of the categories of G-equivariant matrix factorizations. In this case, an isomorphism emerges between the Hochschild cohomology of category $\mathrm{MF}_G(f)$ and the Jacobian algebra $\mathrm{Jac}(\bar{f})$.

In this work, we utilize a theorem introduced in [BP, Io] (Theorem 25 in text), which suggests a method of constructing an orbifold equivalence $(f, G) \sim (\bar{f}, \{\mathrm{id}\})$ via a crepant resolution of $\mathbb{C}^N/G$. Herewith we obtain orbifold equivalences between pairs containing non-invertible polynomials. We use the explicit form of polynomials with a «Loop with Branches» graph to construct a new polynomial corresponding to the original graph with one extra edge (Theorem 26 in the text). In addition, we use Shklyarov's [Sh] techniques to calculate the structure constants of $\mathsf{HH}^*(\mathrm{MF}_G(f))$. We find thereby an explicit isomorphism between their Frobenius algebras (Proposition 34 in text). These results may be used to investigate Frobenius structures and mirror symmetry for (f, G) via the classical singularity structures for $\bar{f}$.

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2. Preliminaries

Consider the ring of polynomials with complex coefficients $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, x_2, \dots, x_N]$. We call a polynomial $f \in \mathbb{C}[\mathbf{x}]$ *non-degenerate* if f has an isolated singularity at the origin; that is, the system of equations $\{\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_N} = 0\}$ has a unique solution at 0. Moreover, throughout this paper we assume that f does not contain summands of the type $x_i x_j$.

Definition 1. A polynomial $f \in \mathbb{C}[\mathbf{x}]$ is called *quasihomogeneous* with a set of weights $(v_1, ..., v_N, d) \in \mathbb{Z}_{\geq 0}^{N+1}$ if

$$f(\lambda^{v_1} x_1, \lambda^{v_2} x_2, ..., \lambda^{v_N} x_N) = \lambda^d f(x_1, x_2, ... x_N)$$

for all $\lambda \in \mathbb{C}^*$.

Remark 2. In Sections 6 and 7, we focus on the reduced system of weights $(q_1, q_2, ... q_N)$, which can be derived as $(q_1, q_2, ... q_N) := (v_1/d, \dots, v_N/d)$, with $q_i \leq \frac{1}{2}$ (see [HK], Theorem 3.7).

Let $f \in \mathbb{C}[\mathbf{x}]$ be a non-degenerate quasihomogeneous polynomial.

Definition 3. The *Jacobian algebra* of a polynomial f is the quotient ring of the ring of polynomials $\mathbb{C}[\mathbf{x}]$ by the ideal generated by partial derivatives of f:

$$\mathrm{Jac}(f) := \mathbb{C}[x_1, x_2, \dots, x_N] / _{\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_N}\right)}$$

The dimension of the Jacobian algebra is called the *Milnor number* and is denoted μ_f. Additionally, $\mu_f < \infty$ if and only if f is non-degenerate (see [AGV]).

Definition 4. A non-degenerate quasihomogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]$ is *invertible* if the following conditions are satisfied:

- The numbers of variables and monomials in f are identical:

$$f(x_1, \dots, x_N) = \sum_{i=1}^{N} c_i \prod_{j=1}^{n} x_j^{E_{ij}},$$

  for $i, j = 1, \dots, N$, with $c_i \in \mathbb{C}^*$ and E_{ij} integer and non-negative.
- The matrix (E_{ij}) is invertible over $\mathbb{Q}$.

If a non-degenerate quasihomogeneous polynomial f does not satisfy these conditions, it is non-invertible: this is the case we dwell on in this work.

Definition 5. The group of *maximal diagonal symmetries* of f is defined by

$$G_f = \{(g_1, g_2, \dots, g_N) \in (\mathbb{C}^*)^N \mid f(g_1 x_1, g_2 x_2, \dots, g_N x_N) = f(x_1, x_2, \dots, x_N)\}$$

Any subgroup $G \subseteq G_f$ is a group of *diagonal symmetries* (or simply a group of *symmetries*). The element $g \in G$ can be associated to the algebra $\mathrm{Jac}(f^g)$ of the polynomial $f^g = f|_{Fix(g)}$, where $Fix(g) = \{x \in \mathbb{C}^N \mid g \cdot x = x\}$. In addition, if $Fix(g) = \{0\}$, we put $f^g := 1$.

Proposition 6 ([ET1], Prop. 5)**.** *If a polynomial f is non-degenerate, then for every $g \in G$ with non-trivial fixed locus, the polynomial f^g is also non-degenerate.*

3. Polynomials and graphs

3.1. **Combinatorial data.** Following [HK], we move forward to introduce the conditions on the weight system of quasihomogeneous f, which render it non-degenerate. For a fixed $N \in \mathbb{N}$ we denote $I := \{1, ..., N\}$, and set e_i to be the standard basis in the lattice $\mathbb{Z}_{\geq 0}^N$. For a subset $J \subseteq I$ and a system of weights $(v_1, ..., v_N, d) \in \mathbb{Z}_{\geq 0}^{N+1}$ with $v_i < d$ and $k \in \mathbb{Z}_{\geq 0}$ we indicate:

$$\begin{aligned} \mathbb{Z}_{\geq 0}^J &:= \{\alpha \in \mathbb{Z}_{\geq 0}^N \mid \alpha_i = 0 \text{ for } i \notin J\}, \\ (\mathbb{Z}_{\geq 0}^N)_k &:= \{\alpha \in \mathbb{Z}_{\geq 0}^N \mid \sum_i \alpha_i \cdot v_i = k\}, \\ (\mathbb{Z}_{\geq 0}^J)_k &:= \mathbb{Z}_{\geq 0}^J \cap (\mathbb{Z}_{\geq 0}^N)_k. \end{aligned}$$

Lemma 7 ([HK], Lemma 2.1.). *Let us fix the system of weights* $(v_1, ..., v_N, d) \in \mathbb{Z}_{\geq 0}^{N+1}$ *with* $v_i < d$ *and a subset* $R \subseteq (\mathbb{Z}_{\geq 0}^N)_d$. *For any* $k \in I$, *we define the set*

$$R_k := \{\alpha \in (\mathbb{Z}_{\geq 0}^N)_{d-v_k} \mid \alpha + e_k \in R\}.$$

The following conditions are pairwise equivalent:

(C1): $\forall\ J \subset I$ *such that* $J \neq \varnothing$
a) $\exists\ \alpha \in R \cap \mathbb{Z}_{\geq 0}^J$
or b) $\exists\ K \subset I \backslash J$ *such that* $|K| = |J|$
and $\forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{Z}_{\geq 0}^J$.

(C1)': *The same as (C1), but only for* J *such that* $|J| \leq \frac{N+1}{2}$.

(C2): $\forall\ J \subset I$ *such that* $J \neq \varnothing$
$\exists\ K \subset I$ *such that* $|K| = |J|$
and $\forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{Z}_{\geq 0}^J$.

(C2)': *The same as (C2), but only for* J *such that* $|J| \leq \frac{N+1}{2}$.

This lemma serves to formulate the criteria for the non-degeneracy of polynomial f from a combinatorial perspective.

Theorem 8 ([HK, Sa, OPSh, KS]). *Let* $(v_1, ..., v_n, d) \in \mathbb{N}^{N+1}$ *be a system of weights with* $v_i < d$ *and for any* $f \in \mathbb{C}[\mathbf{x}]$ *define* $supp(f) = \{\alpha \in \mathbb{Z}_{\geq 0}^N \mid x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$ *is a term of* $f\}$.

(a) Let $f \in \mathbb{C}[\mathbf{x}]$ *be a quasihomogeneous polynomial. Then from the condition*

(IS1): f *is non-degenerate,*

We conclude that the set $R := supp(f) \subset (\mathbb{Z}_{\geq 0}^N)_d$ *satisfies the conditions (C1)-(C2)'.*

(b) Let R *be a subset* $(\mathbb{Z}_{\geq 0}^N)_d$. *Then, the following conditions are equivalent:*

(IS2): *There is a quasihomogeneous polynomial* f *such that* $supp(f) \subseteq R$ *and* f *is non-degenerate.*

(IS2)': *A generic quasihomogeneous polynomial* f *such that* $supp(f) \subseteq R$ *is non-degenerate.*

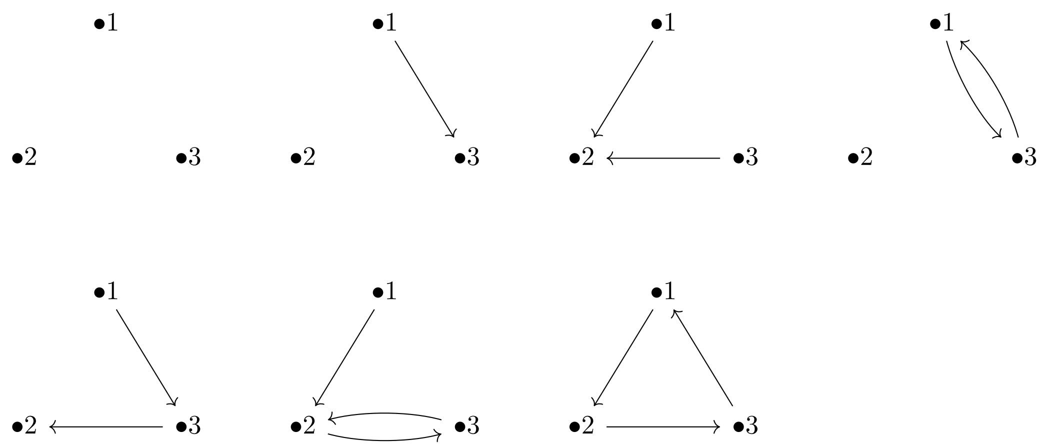
(C1) to (C2)': R *satisfies the conditions (C1)-(C2)'.*

3.2. **Graph description of non-degeneracy.** A map $\kappa\colon I \to I$ is a *choice* if it satisfies the following condition: For every $j \in I$ the sets $J = \{j\}$ and $K = \{\kappa(j)\}$ satisfy $(C2)$ for some fixed R. If we consider any quasihomogeneous polynomial f and put $R = supp(f)$, then κ is a choice if f contains as a summand $b_j \cdot x_j^{a_j} \cdot x_{\kappa(j)}$, where $b_j \in \mathbb{C}^*, a_j \in \mathbb{N}$ and $a_j \geq 2$. As a consequence, polynomial f can be expressed as $f = f_\kappa + f_{add}$, where f_κ is a part determined by the choice κ and $f_{add} := f - f_\kappa$.

Proposition 9 ([HK], Lemma 3.5.). *A polynomial* f *is invertible if and only if* $f = f_\kappa$ *for some* κ. *In other words,* $f_{add} = 0$.

In this paper, we work with non-invertible polynomials (i.e., $f_{add} \neq 0$) using the graph method introduced in [HK]. Using the map $\kappa : I \to I$ we construct the graph Γ_κ with vertices labeled by the set I. Hence, there is an oriented edge pointing from the j-th vertex to the i-th vertex if and only if $i = \kappa(j)$. We assume that if $\kappa(j) = j$ the edge can be skipped, thus we obtain a graph without any self-loops. The conjugacy class κ with respect to the natural action of the symmetric group on the set of indices will be called its *type*. In this case the oriented graph without numbering of vertices defines the type.

Theorem 10 ([AGV], §13.2)**.** *Let $f(x_1, x_2, x_3)$ be a non-degenerate polynomial of three variables. Then, the graphs corresponding to all possible choices for $f(x_1, x_2, x_3)$ are exhausted by the following graphs (types) up to a permutation of variables:*



Polynomials corresponding to the graphs discussed in Theorem 10 have the following form:

- $f_1 = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$
- $f_2 = x_1^{a_1} x_3 + x_2^{a_2} + x_3^{a_3}$
- $f_3 = x_1^{a_1} x_2 + x_3^{a_3} x_2 + x_2^{a_2} + \varepsilon_{1,3} x_1^{b_1} x_3^{b_3}$, where $(a_2 - 1) \mid \mathrm{lcm}(a_1, a_3)$ and $\varepsilon_{1,3} \in \mathbb{C}^*$
- $f_4 = x_2^{a_2} + x_1^{a_1} x_3 + x_3^{a_3} x_1$
- $f_5 = x_1^{a_1} x_3 + x_3^{a_3} x_2 + x_2^{a_2}$
- $f_6 = x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_2 + \varepsilon_{1,3} x_1^{b_1} x_3^{b_3}$, where $(a_2 - 1) \cdot \gcd(a_1, a_3) \mid (a_1 - 1)$ and $\varepsilon_{1,3} \in \mathbb{C}^*$
- $f_7 = x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$

Conditions for polynomials f_3 and f_6 are obtained by writing down the explicit equations on the weights (v_1, v_2, v_3) (see [AGV]). In general, possible graphs are characterized as

Proposition 11 ([HK], Lemma 3.1)**.** *The exact graphs that occur as graphs of maps $\kappa : I \to I$ are those whose components are either globally oriented trees or consist of one globally oriented cycle and finitely many globally oriented trees whose roots are on the cycle.*

In other words, any connected component of Γ_κ is a loop with branches (see Fig. 1), including the case when the "loop" is of length 1. Therefore, any polynomial f_κ can be decomposed as follows:

$$f_\kappa = f_{inv} + f_1 + f_2 + \cdots + f_p,$$

where f_{inv} is an invertible polynomial, and f_i is a polynomial corresponding to the i-th connected component of the graph κ.

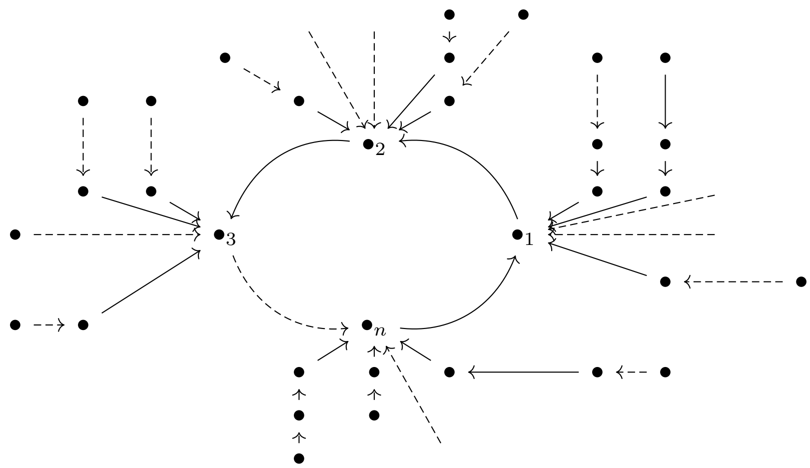


FIGURE 1. «Loop with Branches»-shaped graph

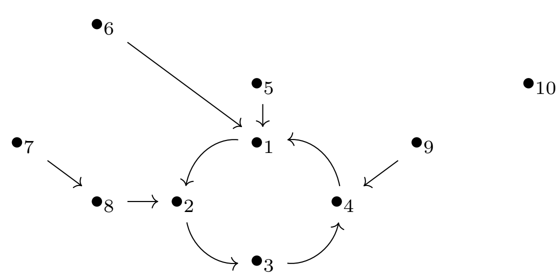
Since we are interested in the non-invertible case, in the following we denote by f_κ a polynomial with a singular connected component arising from the «Loop with Branches» graph and a set of powers $(a_1, \dots, a_N)$ defining the weight system $(v_1, ..., v_N, d) \in \mathbb{Z}_{\geq 0}^{N+1}$.

Please note that every variable corresponds to a node with only one outgoing arrow, hence the claim:

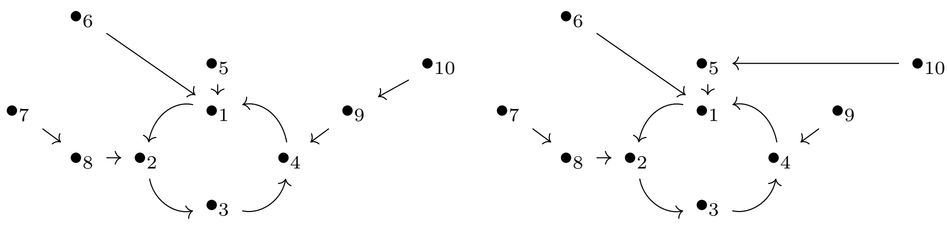
Proposition 12. *Let $f_\kappa = \sum_{j=1}^{N} b_j x_j^{a_j} x_{\kappa(j)}$. Then all b_j can be assumed to be equal to 1.*

Let us also define a gluing operation on the graphs, which attaches a point to a predefined graph. Let $\kappa \colon I \to I$ be a map defining a «Loop with Branches» graph Γ_κ and let $T_\kappa \subset I$ be a set of leaves of Γ_κ, i.e., the set of vertices on the end of each branch. Let $t \in T_\kappa$ be a leaf and $\{N+1\}$ an isolated vertex. We define the operation $\circ_{\Gamma_\kappa}(N+1, t)$ which adds an arrow from the isolated vertex $\{N+1\}$ to the leaf t, meaning that $\circ_{\Gamma_\kappa}(N+1, t) = \Gamma_{\bar{\kappa}}$, where $\bar{\kappa}$ is a new graph with the additional glued vertex. We may also extend the action of $\circ_\kappa$ to act on k isolated vertices and k leaves. In this case we consider $\circ_{\Gamma_\kappa}((N+1, \dots, N+k), (t_1, \dots, t_k))$, which connects $\{N+i\}$ to each t_i. This construction will be particularly useful in Sections 6 and 7 to describe a new polynomial $\bar{f}$ paired with a graph obtained by the action of the gluing operation and demonstrate the construction of orbifold equivalence.

Example 13. Let us consider separately the graph Γ_κ (as shown below) with an isolated vertex $\{10\}$.



Then $\circ_{\Gamma_\kappa}(\{10\},\{9\}) = \Gamma_{\kappa_2}$ and $\circ_{\Gamma_\kappa}(\{10\},\{5\}) = \Gamma_{\kappa_3}$, where Γ_{κ_2} and Γ_{κ_3} are the following graphs:



Γ_{κ_2} Γ_{κ_3}

4. Non-invertible quasihomogeneous singularities

Now we start with a map κ and systems of weights $(v_1, ..., v_N, d) \in (\mathbb{Z}^{N+1}_{\geq 0})_d$, and use them to construct a quasihomogeneous polynomial f_κ (please note that it may be degenerate), and examine $R = supp(f_\kappa)$. In this section our goal is to obtain an explicit non-degenerate polynomial f, such that $f = f_\kappa + f_{add}$ starting from a predetermined f_κ.

Definition 14. A (nonempty) subset of indices $J \subset I = \{1, \dots, N\}$ is a *failing set* for a given R if it does not satisfy the condition $(C1)$ (or any other equivalent condition), for instance:

$$\text{(NC1):}\quad R \cap \mathbb{Z}^J_{\geq 0} = \varnothing \text{ and } \forall\ K \subset I \backslash J \text{ such that } |K| = |J| \text{ it is true that } \exists\ k \in K\ R_k \cap \mathbb{Z}^J_{\geq 0} = \varnothing$$

It means also that R satisfies the conditions of Lemma 7 if there is no failing set for it. Note that if failing sets for R exist, then f_κ must be degenerate according to Theorem 8. We seek to identify a connection between failing sets for such R and the map κ.

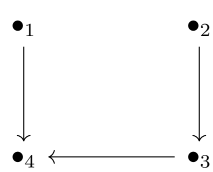
Remark 15. Note that for a given graph Γ_κ a set of powers $(a_1, \dots, a_N)$ defines the system of weights $(v_1, ..., v_N, d) \in (\mathbb{Z}^N_{\geq 0})_d$. Since at times it is less complicated to work with powers, we would apply this property in the future.

Proposition 16. *Let J be a failing set for $R = supp(f_\kappa)$. Then $\forall j \in \{1, 2, \dots, N\}$, $\{j, \kappa(j)\} \nsubseteq J$. Moreover, the same is true for the fixed points of κ: $\kappa(j) = j$.*

Proof. In this case, $a_j e_j + e_{\kappa(j)} \in R \cap \mathbb{Z}^J_{\geq 0} \neq \varnothing$, thus, according to (C1a), J is not failing. □

Let us study an example: finding failing sets for some degenerate quasihomogeneous polynomial f_κ.

Example 17. We start with a map $\kappa : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ which corresponds to the graph



and an arbitrary set of powers (a_1, a_2, a_3, a_4) $(a_i \geq 2)$. Then we have a polynomial $f_\kappa = x_4^{a_4+1} + x_1^{a_1} x_4 + x_3^{a_3} x_4 + x_2^{a_2} x_3$, which is degenerate, and we can consider $R = supp(f_\kappa)$. Let us write a matrix in such a way that R would coincide with the set of its rows:

$$\begin{bmatrix} 0 & 0 & 0 & a_4+1 \\ 0 & 0 & a_3 & 1 \\ 0 & a_2 & 1 & 0 \\ a_1 & 0 & 0 & 1 \end{bmatrix}$$

We would like to find all failing sets for such R. Since κ is a choice, all $J \subset I$ with $|J| = 1$ are not failing sets. Therefore, we move to the case $|J| = 2$ (since according to (C1)' we may only consider sets J with $|J| \le \frac{5}{2} < 3$). Note that $\forall j \in \{1, 2, 3, 4\}$, $J \neq \{j, \kappa(j)\}$ by the proposition above. It is equally important to note that if $4 \in J$, then J is also not a failing set by (C1a), since $(a_4 + 1)e_4 \in R \cap \mathbb{Z}^J_{\ge 0} \neq \varnothing$. Consequently, the only possible failing set here would be $J = \{1, 3\}$ (since for $J = \{1, 2\}$ the set $K = \{3, 4\}$ satisfies the condition (C1)). As $R \cap \mathbb{Z}^J_{\ge 0} = \varnothing$, and the only possible option for which the condition $R_1 \cap \mathbb{Z}^J_{\ge 0} = R_3 \cap \mathbb{Z}^J_{\ge 0} = \varnothing$ follows would still be $K = \{2, 4\}$. Consequently, $J = \{1, 3\}$ is a failing set for R introduced above.

Now a definition is required to present the construction of f_{add}.

Definition 18. Let $(v_1, ..., v_N, d) \in \mathbb{Z}^{N+1}_{\ge 0}$ be a system of weights, let $R \subset (\mathbb{Z}^N_{\ge 0})_d$ a subset of the lattice, and let $F_R \subset 2^I$ a set of all failing sets for R. A collection of sets $A_R := \{J_1, \dots, J_l\}$ with $J_i \subset I$ is said to be *admissible* for R if the following conditions hold:

(i) J_k is a failing set for R for $1 \le k \le l$, i.e., $A_R \subset F_R$

(ii) For any $J \in F_R$ there is an element $J_k \in A_R$ such that $J \backslash J_k$ is not failing.

Now we are ready to introduce the theorem.

Theorem 19. *Let A_R be an admissible collection of $R = supp(f_\kappa)$ and assume there is a set of multipowers $\{b_{J_K} \in (\mathbb{Z}^{J_K}_{\ge 0})^d | J_K \in R$ and $b_s > 0$ for $s \in J_K\}$.*

Then there is a set $\{\epsilon_{J_k} \in \mathbb{C}^*\}$ such that the polynomial $f = f_\kappa + f_{add}$ is non-degenerate with

$$f_{add} = \sum_{J_k \in A_R} \varepsilon_{J_k} x_{k_1}^{b_{k_1}} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}}$$

Before we prove this theorem, let us consider an example that illustrates the construction of f_{add}.

Example 20. Let us turn back to Example 17 and set $(a_1, a_2, a_3, a_4) = (6, 9, 3, 7)$ defining the system of weights $(v_1, v_2, v_3, v_4, d) = (9, 5, 18, 9, 63)$. The only failing set for

$$f_\kappa = x_4^7 + x_1^6 x_4 + x_3^3 x_4 + x_2^9 x_3$$

would then be $J = \{1, 3\}$. So the admissible collection is unique, and we have $A_R = \{J\}$. Note that $b_1 = 3$ and $b_3 = 2$ satisfy the condition $b_1 v_1 + b_3 v_3 = 63$. It follows from the theorem above that the polynomial

$$f = f_\kappa + f_{add} = x_4^7 + x_1^6 x_4 + x_3^3 x_4 + x_2^9 x_3 + x_1^3 x_3^2$$

is non-degenerate, as can be verified by straightforward computation (here $\varepsilon_J = 1$).

In order to prove the theorem, we require a lemma.

Lemma 21. *Let $A_R = \{J_1, \dots, J_l\}$ be admissible for R. Then there is no failing set for the set*

$$R' := R \cup \mathbb{Z}^{J_1}_{\ge 0} \cup \mathbb{Z}^{J_2}_{\ge 0} \cup \dots \cup \mathbb{Z}^{J_l}_{\ge 0} \subset (\mathbb{Z}^N_{\ge 0})_d. \tag{4.1}$$

Proof. Suppose that $J \subset I$ is a failing set for R', but not for R. Since $R \subset R'$, then $(C2a)$ for R cannot be satisfied for J (otherwise J would not be a failing set for R'). Therefore, $(C1b)$ for R must be satisfied and $\exists\ K \subset I$ such that $|K| = |J|$ and $\forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{Z}^J_{\ge 0}$. However, since $R_k \subset R'_k$ we also have $\alpha \in R'_k \cap \mathbb{Z}^J_{\ge 0}$. Consequently, (C1b) is also satisfied for R', and thus J is not a failing set for R'.

Suppose now that J is a failing set for R. By the definition of an admissible collection, $\exists\ J_k \in A_R$ such that $J = J_k \sqcup J_k^c$, where J_k^c is the complementary set. It means that $\mathbb{Z}^{J_k}_{\ge 0} \neq \varnothing$ and in this case we have

$$\mathbb{Z}^J_{\ge 0} \cap R' = \mathbb{Z}^J_{\ge 0} \cap (R \cup \dots \cup \mathbb{Z}^{J_k}_{\ge 0}) \supset \mathbb{Z}^{J_k}_{\ge 0} \neq \varnothing,$$

and from (C1a), it follows that J is not a failing set for R', which completes the proof. $\square$

Finally, we may proceed to prove Theorem 19.

Proof. We apply the small lemma above to prove the theorem. Note that

$$R' = supp(f_\kappa) \cup supp(f_{add})$$

for arbitrary non-zero coefficients in f_{add}. By the condition $\sum_{s \in J_k} b_s v_s = d$ and by $(IS2)'$, we have a generic non-degenerate polynomial f with support R; that is, f is a linear combination of monomials from f_κ and f_{add} with some non-zero coefficients. By Proposition 12, we may assume the coefficients of f_κ to be equal to one; by these linear transformations we obtain the sought for set $\{\epsilon_{J_k} \in \mathbb{C}^*\}$. □

5. Admissible collections for the loop with branches

In this section, we employ a polynomial f_κ to present a method of constructing an admissible collection A_R for $R = supp(f_\kappa)$. This gives us a recipe for taking any quasihomogeneous polynomial f_κ from a «Loop with Branches» graph, and turning it into a non-degenerate quasihomogeneous polynomial.

Suppose we have a «Loop with Branches» graph on n vertices (i.e., the length of the loop is n). For any $m \in \{1, \ldots, n\}$ consider the set $S_m \subset \{1, \ldots, N\}$ containing all vertices emitting an arrow pointing to m. We then define $A^m := \{J_l \subset S_m \mid |J_l| \geq 2\}$ as the set of all subsets of S_m with at least two elements.

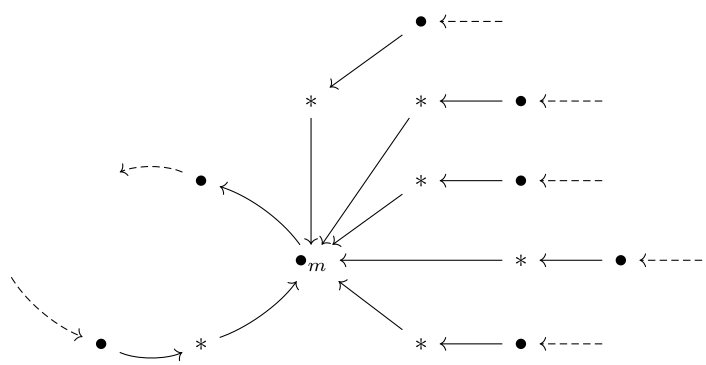


Figure 2. The elements of S_m are marked with $*$

We are now prepared to formulate the Theorem of this Section.

Theorem 22. *The set $A_R = \cup_{m=1}^n A^m$ is an admissible collection for $R = supp(f_\kappa)$.*

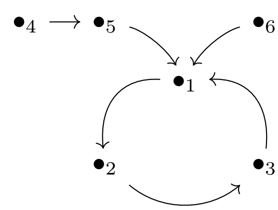
Proof. First of all, any $J_l \in A_R$ must be a failing set for R. Otherwise, by (C1) there is a set $K \subset I$ such that $|K| = |J_l|$ and for all $k \in K$ there is $\alpha_k \in R_k \cap \mathbb{Z}^{J_l}_{\geq 0}$ (since $R \cap \mathbb{Z}^{J_l}_{\geq 0} = 0$). This means that for all $k \in K$, $\alpha_k + e_k \in R$ and $\alpha_k \in \mathbb{Z}^{J_l}_{\geq 0}$, but $R = supp(f_\kappa)$ consists of elements of $\mathbb{Z}^N_{\geq 0}$ with only two non-zero coordinates (and one of them is equal to one). Hence we conclude that α_k should either also have two non-zero coordinates (with one of them equal to one) or only have one. The first case is not possible since it would mean an arrow between some $j_1, j_2 \in J_l$, and the second case is equally not possible since it would mean that $K = \{k\}$, with $\kappa(j_1) = \kappa(j_2) = \cdots = k$, is the unique set satisfying the required condition (C1) and $|K| = 1 < |J_l|$. From this we must conclude that any $J_k \in A^m \subset A_R$ is a failing set for R for any m.

Now we seek to demonstrate that for any failing set J there is a set $J_l \in A_R$ such that $J_l \subset J$. From what we managed to show above, we are able to conclude that any failing set for R has a decomposition $J = U_J^1 \sqcup U_J^2 \sqcup \dots U_J^n \sqcup U_J^c$, where U_J^m contains all indices from I, whose image under κ is equal to $m \in \{1, \dots, n\}$, and U_J^c is the complementary set. Since J is a failing set, there exists an element s such that $|U_J^s| \geq 2$, which means that there is also a set $J_l \in A_R$ such that $U_J^s = J_l$. Consequently, $J_l \subset J$. We demonstrated how for any failing set J there is a set $J_l \in A_R$ such that $J_l \subset J$, which proves that the collection A_R is admissible for R. □

Corollary 23. *Let A_R be a collection as introduced above. For f_{add} defined in Theorem 19, $f = f_\kappa + f_{add}$ is non-degenerate.*

Proof. Follows directly from Theorem 8. □

Example 24. Consider the map κ which gives the graph



and a set of powers $(a_1, a_2, a_3, a_4, a_5, a_6) = (3, 2, 4, 2, 3, 4)$ that defines the weight system

$$(v_1, v_2, v_3, v_4, v_5, v_6, d) = (1, 2, 1, 1, 2, 1, 5);$$

that is, the polynomial defined by κ is of the form

$$f_\kappa = x_1^3 x_2 + x_2^2 x_3 + x_3^4 x_1 + x_5^2 x_1 + x_4^3 x_5 + x_6^4 x_1$$

and $R = supp(f_\kappa)$. By the theorem proven earlier, the admissible collection A_R is of the form $A_R = A^1 = \{\{3, 5\}, \{3, 6\}, \{5, 6\}, \{3, 5, 6\}\}$. Thus a set of corresponding coefficients ε_{J_k} exists, and we write:

$$f_{add} = \varepsilon_{3,5} x_3 x_5^2 + \varepsilon_{3,6} x_3^2 x_6^3 + \varepsilon_{5,6} x_5^2 x_6 + \varepsilon_{3,5,6} x_3^2 x_5 x_6$$

$$f = f_\kappa + f_{add} = x_1^3 x_2 + x_2^2 x_3 + x_3^4 x_1 + x_5^2 x_1 + x_4^3 x_5 + x_6^4 x_1 + \varepsilon_{3,5} x_3 x_5^2 + \varepsilon_{3,6} x_3^2 x_6^3 + \varepsilon_{5,6} x_5^2 x_6 + \varepsilon_{3,5,6} x_3^2 x_5 x_6$$

where, by Theorem 19, the polynomial f is non-degenerate, which could be verified by rigorous calculation (in this case we may set $\varepsilon_{J_k} = 1$ for all $J_l \in A_R$).

6. Crepant resolutions and orbifold equivalence

In this section, we consider a Landau-Ginzburg orbifold (f, G), where the quasihomogeneous polynomial f corresponds to a «Loop with Branches» graph with one isolated vertex, and the group G is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We introduce a new polynomial $\bar{f}$, obtained from f via the gluing operation, and present an orbifold equivalence $(\bar{f}, \{\mathrm{id}\}) \sim (f, G)$.

6.1. **Background.** Let $f \colon \mathbb{C}^N \to \mathbb{C}$ be a holomorphic G-invariant function. Hence, it may be defined as a function on $\mathbb{C}^N/G$. Let $\tau \colon \widehat{\mathbb{C}^N/G} \to \mathbb{C}^N/G$ be a crepant resolution of the singularity. The composition of f with τ defines a function on $\widehat{\mathbb{C}^N/G}$, denoted by $\hat{f} \colon \widehat{\mathbb{C}^N/G} \to \mathbb{C}$. Let $\widehat{\mathbb{C}^N/G}$ be covered by some charts $U_1, \dots, U_s$, all of them isomorphic to $\mathbb{C}^N$; $\hat{f}_i$ denotes the restriction of $\hat{f}$ on each chart

$$\hat{f}_i = \hat{f}|_{U_i} \colon U_i \to \mathbb{C}.$$

Considering a non-degenerate quasihomogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]$ and a group of symmetries G, we can associate the category $\mathrm{MF}_G(f)$ to the pair (f, G) (see [Orl1], [Orl2], [Io] for details). We may reformulate the definition of orbifold equivalence (see [Io]) in terms of categories, giving rise to the following theorem:

Theorem 25. *([BP], [Io]) Suppose that f and $\hat{f}$ both have isolated singularities at the unique points $v \in \mathbb{C}^N/G$ and $w \in \widehat{\mathbb{C}^N/G}$ respectively; then the pairs (f, G) and $(\hat{f}, \{\mathrm{id}\})$ are orbifold equivalent. Namely, there is an equivalence of categories*

$$\mathrm{MF}(\hat{f}) \cong \mathrm{MF}_G(f).$$

Moreover, suppose that $\hat{f}$ has a singularity at the origin and there is an affine chart U_i such that $\hat{f}_i : \mathbb{C}^N \cong U_i \to \mathbb{C}$ has a singularity at 0 and $\hat{f}_j : \mathbb{C}^N \cong U_j \to \mathbb{C}$ does not have any singularities $\forall j \neq i$. Then,

$$\mathrm{MF}(\hat{f}) \cong \mathrm{MF}(\hat{f}_i).$$

This theorem asserts that to establish an orbifold equivalence between (f, G) and $(\bar{f}, \{\mathrm{id}\})$, it is sufficient to find a suitable affine chart for which $\bar{f} = \hat{f}_m$, with $\hat{f}_m$ being the unique non-degenerate polynomial in the collection $\{\hat{f}_i\}$.

6.2. **Orbifold equivalence for the loop with branches.** We consider a polynomial f_{κ_0} with one connected component, and assume that there is an index $t \in T_{\kappa_0}$ (i.e., the set of leaves), such that its corresponding power is even; that is, there is a monomial $x_t^{2a_t} x_{\kappa(t)}$. Without loss of generality, we may set $t = 1$ and $\kappa(1) = 2$.

By Theorem 22 we get $f_0 = f_{\kappa_0} + f_{add}$, where f_{add} is given by the admissible collection for $R = supp(f_{\kappa_0})$. Then, let us investigate the polynomial $f_\kappa = f_{\kappa_0} + x_{N+1}^2$ with the set of powers $(2a_1, \dots, a_N, 2)$, which is quasihomogeneous with a reduced set of weights $(q_1, \dots, q_N, 1/2)$. Its corresponding graph Γ_κ is the disjoint union of Γ_{κ_0} and one isolated vertex. Since x_{N+1}^2 is invertible, the admissible collection A_R for $R = supp(f_\kappa)$ coincides with the admissible collection for $R = supp(f_{\kappa_0})$. As a consequence, $f = f_\kappa + f_{add}$ is non-degenerate by Theorem 22 (with the same f_{add} as for f_0).

The focus of our investigation is the Landau-Ginzburg orbifold (f, G) with symmetry group $G \cong \mathbb{Z}/2\mathbb{Z} \cong \langle g \rangle$ acting as follows:

$$g \cdot x_m = -x_m \text{ if } m = 1; N+1$$

$$g \cdot x_m = x_m \text{ otherwise}$$

This means that g acts non-trivially only on either an isolated vertex or on a leaf with even power.

For the group G discussed earlier, we may write $\mathbb{C}^{N+1}/G \cong \mathbb{C}^{N-1} \times \{w^2 = uv\} \subset \mathbb{C}^{N+1}$ by identifying $\{u = x_1^2, v = x_{N+1}^2, w = x_1 x_{N+1}\}$ or $\{v = x_1^2, u = x_{N+1}^2, w = x_1 x_{N+1}\}$, which grants us two charts U_1 and U_2 covering $\mathbb{C}^{N+1}/G$; here for U_1:

$$(x_2, \dots, x_N, y, z) \longrightarrow (x_2, \dots, x_N, y, yz^2),$$

and for U_2:

$$(x_2, \dots, x_N, y, z) \longrightarrow (x_2, \dots, x_N, y^2 z, z).$$

We are finally prepared to formulate the main theorem of this section:

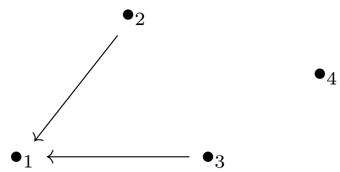
Theorem 26. *Let (f, G) be a Landau-Ginzburg orbifold as above. Then there is a non-degenerate quasihomogeneous polynomial $\bar{f}$ with the following properties:*

(i) *$\bar{f}$ has a reduced system of weights $(2q_1, q_2, \dots, q_N, 1/2 - q_1)$*
(ii) *$\bar{f} = f_{\bar{\kappa}} + \bar{f}_{add}$ with the graph $\Gamma_{\bar{\kappa}} = \circ_{\Gamma_\kappa}(N+1, 1)$ and $\bar{f}_{add} = f_{add}(t_1, \dots, t_N)$ where we put $t_1^2 = x_1$ and $t_m = x_m$ for $m \neq 1$;*

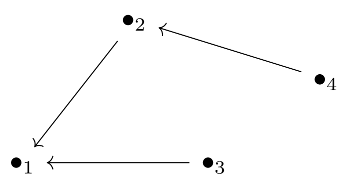
such that (f, G) and $(\bar{f}, \{\mathrm{id}\})$ are orbifold equivalent. In particular, there is an equivalence of categories

$$\mathrm{MF}(\bar{f}) \cong \mathrm{MF}_G(f).$$

Example 27. Consider the polynomial $f_{\kappa_0} = x_1^3 + x_2^4 x_1 + x_3^8 x_1$. Using the earlier conclusions, we get $f = x_1^3 + x_2^4 x_1 + x_3^8 x_1 + x_2^4 x_3^4 + x_4^2$ with $f_{add} = x_2^4 x_3^4$, the reduced system of weights $(1/3, 1/6, 1/12, 1/2)$, and the graph



Consider now the group $G \cong \mathbb{Z}/2\mathbb{Z} \cong \langle g \rangle$, where $g(x_1, x_2, x_3, x_4) = (x_1, -x_2, x_3, -x_4)$; herewith we obtain $\bar{f} = x_1^3 + x_2^2 x_1 + x_3^8 x_1 + x_2^2 x_3^4 + x_4^2 x_3$ with $\bar{f}_{add} = x_2^2 x_4^4$, the reduced system of weights $(1/3, 1/3, 1/12, 1/3)$ and the graph



such that there is an orbifold equivalence $(f, \mathbb{Z}/2\mathbb{Z}) \sim (\bar{f}, \{\mathrm{id}\})$.

7. Proof of Theorem 26

We seek to show that the polynomial $\bar{f}$, discussed in Theorem 26, coincides with $\hat{f}_1$, obtained via the crepant resolution. First, we prove that $\hat{f}_1$ is non-degenerate (i.e., quasihomogeneous and with an isolated singularity at the origin) and possesses the properties stated in Theorem 26. Next, we demonstrate that $\hat{f}_2$ does not have any singularities. Lastly, by Theorem 25, the corresponding Landau–Ginzburg orbifolds (f, G) and $(\hat{f}_1, \{\mathrm{id}\})$ are orbifold equivalent, completing the proof.

Let us rewrite $\hat{f}_1$ in explicit form. Following the decomposition of f, we get

$$\hat{f}_1 = (\hat{f}_\kappa)_1 + (\hat{f}_{add})_1.$$

Recall that f_0 is a polynomial with a «Loop with Branches» graph κ_0 discussed in Section 6. Then

$$(\hat{f}_\kappa)_1 = f_{\kappa_0}(y, x_2, \ldots, x_N) + z^2 y \tag{7.1}$$

Since $G \cong \mathbb{Z}/2\mathbb{Z}$ is a group of symmetries, each term of f_{add} is of even degree in x_1. Explicitly,

$$f_{add} = \sum_{\{J_r \in A_R \mid 1 \notin J_r\}} \varepsilon_{J_r} x_{s_1}^{b_{s_1}} x_{s_2}^{b_{s_2}} \ldots x_{s_l}^{b_{s_l}} + \sum_{\{J_r \in A_R \mid 1 \in J_r\}} \varepsilon_{J_r} x_1^{2b_1} x_{s_2}^{b_{s_2}} \ldots x_{s_l}^{b_{s_l}}$$

is to be applied to the admissible collection A_R and $\varepsilon_{J_r} \in \mathbb{C}^*$. After a change of variables:

$$f_{add} = \sum_{\{J_r \in A_R \mid 1 \notin J_r\}} \varepsilon_{J_r} x_{s_1}^{b_{s_1}} x_{s_2}^{b_{s_2}} \ldots x_{s_l}^{b_{s_l}} + \sum_{\{J_r \in A_R \mid 1 \in J_r\}} \varepsilon_{J_r} y^{b_1} x_{s_2}^{b_{s_2}} \ldots x_{s_l}^{b_{s_l}}$$

Proposition 28. *The polynomial $\hat{f}_1$ is quasihomogeneous with the reduced system of weights $(2q_1, q_2, \ldots, q_N, 1/2 - q_1)$, where $(q_1, \ldots, q_N)$ are weights of f_0.*

Proof. The reduced weights are determined by the system of equations $a_i q_i + q_{\kappa(i)} = 1$, where a_i are fixed exponents of the polynomial, and $q_{\kappa(i)} = 0$ iff $\kappa(i) = i$ (in particular, $q_{N+1} = 1/2$). Recall that f has the set of powers $(2a_1, a_2, \ldots, a_N, 2)$ and let $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{N+1})$ be the new powers following resolution and restriction to the first chart. It is not hard to notice that if

$s \neq 1, N+1$, then $a_s = \hat{a}_s$. Therefore, equations $\hat{a}_s q_s + q_{\kappa(s)} = 1$ hold, which means that $\hat{q}_s = q_s$. Now, if $s = 1$, we get $\hat{a}_1 = 2a_1/2 = a_1$, and in this case the equation

$$\hat{a}_1(2q_1) + \hat{q}_2 = a_1(2q_1) + q_2 = 2a_1q_1 + q_2 = 1$$

holds; we conclude that $\hat{q}_1 = 2q_1$. Similarly, for $s = N+1$ we know that $\hat{a}_{N+1} = 2$ according to the explicit form of $\hat{f}_1$, and we only need to solve the following system of equations:

$$\begin{aligned}\hat{a}_{N+1}\hat{q}_{N+1} + \hat{q}_1 &= 1\\ 2\hat{q}_{N+1} + 2q_1 &= 1\\ \hat{q}_{N+1} &= \frac{1-2q_1}{2}\end{aligned}$$

Here, the condition $\sum_{s\in J_k} b_s q_s = 1$ holds, since for each choice of J_k we only modify the term with $s = 1$, and $\hat{b}_{k_1} = 2b_{k_1}/2 = b_{k_1}$, which implies $2b_{k_1}q_1 = \hat{b}_{k_1}2q_1 = \hat{b}_{k_1}\hat{q}_1$. □

Corollary 29. *Since $\hat{f}_1$ is quasihomogeneous, it must correspond to some graph. Using the explicit form (7.1), we conclude that the graph in question is $\Gamma_{\bar{\kappa}} = \circ_{\Gamma_\kappa}(N+1, 1)$.*

Since we managed to prove that $\hat{f}_1$ is quasihomogeneous, the last statement that we need to check is:

Proposition 30. *All critical points of $\hat{f}$ are contained in the chart U_1, and $\hat{f}_1$ is non-degenerate.*

Proof. Our objective is to show that 0 is the only isolated solution of the system $\{d\hat{f}_s = 0\}$ for $s = 1$. Recall the explicit form of $\hat{f}_1$, then set $y = x_1$ and $z = x_{N+1}$:

$$\hat{f}_1 = (\hat{f}_\kappa)_1 + (\hat{f}_{add})_1 = f_{\kappa_0}(x_1,\dots,x_N) + x_{N+1}^2 x_1 + \sum_{\{J_r\in A_R \mid 1\notin J_r\}} \varepsilon_{J_r} x_{s_1}^{b_{s_1}} x_{s_2}^{b_{s_2}} \dots x_{s_l}^{b_{s_l}} + \sum_{\{J_r\in A_R \mid 1\in J_r\}} \varepsilon_{J_r} x_1^{b_1} x_{s_2}^{b_{s_2}} \dots x_{s_l}^{b_{s_l}}$$

It is easy to deduce that if s satisfies $1 < s < N+1$ and does not belong to any set in A_R, then

$$\frac{\partial \hat{f}_1}{\partial x_s} = \frac{\partial f_{\kappa_0}}{\partial x_s}.$$

Therefore, since f is non-degenerate, 0 is a solution of the above system.

If s such that $1 < s < N+1$ and $s \in J_k$ for some $J_k \in A_R$, then the system of equations

$$\frac{\partial \hat{f}_1}{\partial x_s} = a_s x_s^{a_s-1} t_{\kappa(s)} + \sum_{w\in\kappa^{-1}(s)} x_w^{a_w} + \sum_{J_k\in A_R\wedge s\in J_k} \varepsilon_{J_k} b_s x_s^{b_s-1} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}}$$

obviously vanishes at 0. Likewise, for $s = 1, N+1$, 0 is a solution:

$$\frac{\partial \hat{f}_1}{\partial x_1} = a_1 x_1^{a_1-1} x_2 + x_{N+1}^2 + \sum_{\{J_r\in A_R \mid 1\in J_r\}} \varepsilon_{J_r} 2b_1 x_1^{2b_1-1} x_{s_2}^{b_{s_2}} \dots x_{s_l}^{b_{s_l}} = 0;$$

$$\frac{\partial \hat{f}_1}{\partial x_{N+1}} = x_1 x_{N+1} = 0.$$

We have demonstrated that 0 is a solution of the equations above. Note also that the change of variables $\{t_1 = x_1, t_{N+1}^2 = x_1 x_{N+1}^2, t_1 t_{N+1} = x_1 x_{N+1}\}$ is a diffeomorphism, and since f is non-degenerate, so is $\hat{f}_1$.

Let us come back to $\hat{f}_2$ to rewrite it similarly to $\hat{f}_1$ earlier; consider now the two equations:

$$\frac{\partial \hat{f}_2}{\partial x_1} = 2a_1 x_1^{2a_1-1} x_{N+1}^{a_{N+1}} x_2 + \sum_{J_k\in A_R\wedge 1\in J_k} \varepsilon_{J_k} 2b_{k_1} x_1^{2b_{k_1}-1} x_{N+1}^{b_{k_1}} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}} = 0,$$

$$\frac{\partial \hat{f}_2}{\partial x_{N+1}} = a_i x_1^{2a_1} x_{N+1}^{a_1-1} x_2 + \sum_{J_k \in A_R \wedge 1 \in J_k} \varepsilon_{J_k} b_{k_1} x_1^{2b_{k_1}} x_{N+1}^{b_{k_1}-1} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}} + 1 = 0.$$

Clearly, these equations have no solution at the origin. Moreover, the left-hand side of the second equation does not vanish when $x_1 = 0$ or $x_{N+1} = 0$. This implies that we must have $x_1 \neq 0$ and $x_{N+1} \neq 0$. We then divide the first equation by $2x_1^{2a_1-1} x_{N+1}^{a_1}$ and the second one by $x_1^{2a_1} x_{N+1}^{a_1-1}$, deducing that

$$a_1 x_2 + \sum_{J_k \in A_R \wedge 1 \in J_k} \varepsilon_{J_k} b_{k_1} x_1^{2b_{k_1}-2a_1} x_{N+1}^{b_{k_1}-a_1} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}} = 0,$$

$$a_1 x_2 + \sum_{J_k \in A_R \wedge 1 \in J_k} \varepsilon_{J_k} b_{k_1} x_1^{2b_{k_1}-2a_1} x_{N+1}^{b_{k_1}-a_1} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}} + x_1^{-2a_1} x_{N+1}^{1-a_1} = 0,$$

which implies $x_1^{-2a_1} x_{N+1}^{1-a_1} = 0$. Thus, the system $\{d\hat{f}_2 = 0\}$ has no solution, which concludes the proof. □

Remark 31. Since orbifold equivalence is an equivalence relation, this result extends to the case of k isolated points, with $G = \underbrace{\mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}}_{k}$, by systematically applying Theorem 26.

8. Isomorphism of algebras

The equivalence of categories signals the existence of an isomorphism between the corresponding Frobenius algebras. In this section, we construct this isomorphism explicitly.

We associate to a Landau-Ginzburg orbifold (f, G), the Hochschild cohomology $\mathsf{HH}^*(MF_G(f))$ of the category of G-equivariant matrix factorizations. In order to manipulate this ring, we construct an algebra $\mathrm{Jac}(f, G)$ in such a way that $\mathrm{Jac}(f, G) \cong \mathsf{HH}^*(MF_G(f)$ (see [Sh], [BTW1], [BT1] for details). To define $\mathrm{Jac}(f, G)$ let us introduce a vector space

$$\mathrm{Jac}'(f, G) = \bigoplus_{g \in G} \mathrm{Jac}(f^g)\xi_g,$$

where ξ_g are formal generators associated with $g \in G$. We then define multiplication in this vector space (see [Sh] for details).

Definition 32. Let $\theta_1, \theta_2, \dots, \theta_N$ and $\partial_{\theta_1}, \partial_{\theta_2}, \dots, \partial_{\theta_N}$ be formal variables. Then the *Clifford algebra* Cl_N is the factor-algebra of $\mathbb{C}[\theta_1, \theta_2, \dots, \theta_N, \partial_{\theta_1}, \partial_{\theta_2}, \dots, \partial_{\theta_N}]$ by the relations:

$$\theta_i \theta_j = -\theta_j \theta_i$$
$$\partial_{\theta_i} \partial_{\theta_j} = -\partial_{\theta_j} \partial_{\theta_i}$$
$$\partial_{\theta_i} \theta_j = \delta_{ij} - \theta_j \partial_{\theta_i}$$

For $I \subseteq \{1, \dots, n\}$ we use the following notations:

$$\partial_{\theta_I} := \prod_{i \in I} \partial_{\theta_i}, \quad \theta_I := \prod_{i \in I} \theta_i,$$

where indices are written in an increasing order. We also introduce the following notations for the Cl_N-modules:

$$\mathbb{C}[\theta] := Cl_N/_{Cl_N\langle \partial_{\theta_1}, \dots, \partial_{\theta_N}\rangle}, \quad \mathbb{C}[\partial_\theta] := Cl/_{Cl_N\langle \theta_1, \dots, \theta_N\rangle}$$

Definition 33. The map $\nabla_i^{x \to (x,y)} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}, \mathbf{y}]$ given by

$$\nabla_i^{x \to (x,y)}(f(x)) = \frac{f(y_1, y_2, \dots, y_{i-1}, x_i, x_{i+1}, \dots, x_N) - f(y_1, y_2, \dots, y_i, x_{i+1}, x_{i+2}, \dots, x_N)}{x_i - y_i}$$

is the i-th *difference derivative* of $f(x)$.

Now we may look into the structure constants of multiplication in $\mathrm{Jac}'(f,G)$. For $g \in G$ denote $I_g := \{i | g_i = 1\}$, $I_g^c := \{1,\dots,N\} \setminus I_g$ and $d_g := |I_g^c|$. For each pair $(g,h) \in G \times G$ we write $d_{g,h} := \frac{1}{2}(d_g + d_h - d_{gh})$, then we define $\sigma_{g,h} \in \mathrm{Jac}(f^{gh})$ as follows:

- If $d_{g,h} \notin \mathbb{Z}_{\geq 0}$, then $\sigma_{g,h} = 0$.
- If $d_{g,h} \in \mathbb{Z}_{\geq 0}$, then we define $\sigma_{g,h}$ as the coefficient before $\partial_{\theta_{I^c_{gh}}}$ in the following expression expansion:

$$\frac{1}{d_{g,h}!}\Upsilon\left(\left(\lfloor \mathrm{H}_f(x,g(x),x)\rfloor_{gh} + \lfloor \mathrm{H}_{f,g}(x)\rfloor_{gh} \otimes 1 + 1 \otimes \lfloor \mathrm{H}_{f,h}(g(x))\rfloor_{gh}\right)^{d_{g,h}} \otimes \partial_{\theta_{I_g^c}} \otimes \partial_{\theta_{I_h^c}}\right)$$

where

(1) $\mathrm{H}_f(x,g(x),x)$ is an element of $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\theta]^{\otimes 2}$ defined by

$$\mathrm{H}_f(\mathbf{x},\mathbf{y},\mathbf{z}) := \sum_{1\leq j\leq i\leq n} \nabla_j^{\mathbf{y}\to(\mathbf{y},\mathbf{z})}\nabla_i^{\mathbf{x}\to(\mathbf{x},\mathbf{y})}(f)\,\theta_i \otimes \theta_j;$$

$$\mathrm{H}_f(x,g(x),x) = \mathrm{H}_f(\mathbf{x},\mathbf{y},\mathbf{z})|_{\{\mathbf{y}=g(\mathbf{x}),\,\mathbf{z}=\mathbf{x}\}}$$

(2) $\mathrm{H}_{f,g}(\mathbf{x})$ is an element of the module $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\theta]$ given by

$$\mathrm{H}_{f,g}(\mathbf{x}) := \sum_{i,j\in I_g^c,\; j<i} \frac{1}{1-g_j}\nabla_j^{\mathbf{x}\to(\mathbf{x},\mathbf{x}^g)}\nabla_i^{\mathbf{x}\to(\mathbf{x},g(\mathbf{x}))}(f)\,\theta_j\,\theta_i,$$

where $\mathbf{x}^g$ is $(\mathbf{x}^g)_i = x_i$ for $i \in I_g$ and $(\mathbf{x}^g)_i = 0$ for $i \in I_g^c$;

(3) $\lfloor - \rfloor_{gh} : \mathbb{C}[\mathbf{x}] \otimes V \longrightarrow \mathrm{Jac}(f^{gh}) \otimes V$ for $V = \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\theta]^{\otimes 2}$ or $V = \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\theta]$ is a $\mathbb{C}$-linear extension of the map $\mathbb{C}[\mathbf{x}] \longrightarrow \mathrm{Jac}(f^{gh})$;

(4) The degree $d_{g,h}$ is calculated with respect to the natural multiplication defined on $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\theta] \otimes \mathbb{C}[\theta]$;

(5) Υ is the $\mathbb{C}[\mathbf{x}]$-linear extension of the map $\mathbb{C}[\theta]^{\otimes 2} \otimes \mathbb{C}[\partial_\theta]^{\otimes 2} \to \mathbb{C}[\partial_\theta]$, defined by

$$p_1(\theta) \otimes p_2(\theta) \otimes q_1(\partial_\theta) \otimes q_2(\partial_\theta) \mapsto (-1)^{|q_1||p_2|} p_1(q_1) \cdot p_2(q_2),$$

where $p_i(q_i)$ is the action of $p_i(\theta)$ on $q_i(\partial_\theta)$ according to the multiplication structure on the Clifford algebra introduced earlier.

We define the multiplication in $\mathrm{Jac}'(f,G)$ in the following manner:

$$[\phi(\mathbf{x})]\xi_g \cup [\psi(\mathbf{x})]\xi_h = [\phi(\mathbf{x})\psi(\mathbf{x})\sigma_{g,h}]\xi_{gh}, \quad \phi(\mathbf{x}),\psi(\mathbf{x}) \in \mathbb{C}[\mathbf{x}].$$

We will also endow $\mathrm{Jac}'(f,G)$ with an action: for $h \in G$ and $[\phi(x)] \in \mathrm{Jac}(f^g)$

$$h = (h_1,\dots,h_n): \quad [\phi(x)]\xi_g \mapsto \prod_{i\in I_g^c} h_i^{-1} \cdot [\phi(h(x))]\xi_g, \quad [\phi(x)] \in \mathrm{Jac}(f^g).$$

Then we define $\mathrm{Jac}(f,G) = (\mathrm{Jac}'(f,G))^G$. Applying Theorem 26, we get

$$\mathrm{Jac}(\bar f,\{id\}) \cong \mathrm{Jac}(\bar f) \quad \mathrm{Jac}(f,G) \cong (\mathrm{Jac}(f)\xi_{id})^G \oplus \mathrm{Jac}(f^g)\xi_g.$$

Proposition 34. *There is an isomorphism of algebras* $\psi\colon \mathrm{Jac}(\bar f) \xrightarrow{\sim} \mathrm{Jac}(f,G)$*, such that*

$$\psi([x_i]) = \begin{cases} [x_i]\xi_{id}, \text{ if } i \neq 1, N+1 \\ [x_1^2]\xi_{id}, \text{ if } i = 1 \\ \xi_g, \text{ if } i = N+1 \end{cases}$$

Proof. Let us demonstrate that $\mathrm{Jac}(\bar f)$ can be decomposed into the sum of two vector spaces:

$$\mathrm{Jac}(\bar f) = \mathcal{B}_1 \oplus \mathcal{B}_2,$$

with $\mathcal{B}_1 \cong (\mathrm{Jac}(f))^G$ as algebras and $\mathcal{B}_2 \cong \mathrm{Jac}(f^g)x_{N+1}$ as vector spaces. We may now construct a basis of $\mathcal{B}_1$; let $[x_1^{\alpha_1}x_2^{\alpha_2}\dots x_{N+1}^{\alpha_{N+1}}]$ be a basis element of $\mathrm{Jac}(\bar f)$. Note that if $\alpha_1 \neq 0$ and

$\alpha_{N+1} = 0$ (i.e., this element does not depend on variable x_{N+1}), then $[x_1^{2\alpha_1} x_2^{\alpha_2} \dots x_N^{\alpha_N}]$ could be chosen as a basis element of $\mathrm{Jac}(f)$ by the construction of $\bar{f}$. Moreover, the multiplication of elements of the form $[x_1^{2\alpha_1} x_2^{\alpha_2} \dots x_N^{\alpha_N}]$ in $\mathrm{Jac}(\bar{f})$ coincides with multiplication in $(\mathrm{Jac}(f)\xi_{id})^G$ due to invariance under the G-action. Thus, we obtain a basis for the algebra $(\mathrm{Jac}(f)\xi_{id})^G$.

In order to construct a basis for $\mathcal{B}_2$, we may now consider basis elements of $\mathrm{Jac}(\bar{f})$ for which $\alpha_{N+1} \neq 0$. Then α_{N+1} should be equal to 1 and $\alpha_1 = 0$, because $[\frac{\partial \bar{f}}{\partial x_{N+1}}] = [2x_1 x_{N+1}] = [0]$. Thus we get $[x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{N+1}^{\alpha_{N+1}}] = [x_2^{\alpha_2} \dots x_N^{\alpha_N}][x_{N+1}]$ with the first factor lying in $\mathrm{Jac}(f^g)$, which gives the sought for isomorphism.

The last piece that we need to prove is $[x_{N+1}]^2 = (\xi_g)^2$. We calculate $\sigma_{g,g^{-1}}$ using the formula derived earlier. Note that

$$H_{f,g}(x) = \frac{1}{2} \nabla_1^{\mathbf{x}\to(\mathbf{x},\mathbf{x}^g)} \nabla_{N+1}^{\mathbf{x}\to(\mathbf{x},g(\mathbf{x}))}(f)\, \theta_1\, \theta_{N+1} = 0,$$

since f does not have a summand containing both variables x_1 and x_{N+1}. Similarly, $H_{f,g^{-1}}(g(x)) = 0$. Then $\sigma_{g,g^{-1}}$ is the coefficient of 1 in

$$\frac{1}{2} \Upsilon([\mathrm{H}_f(x, g(x), x)]^2 \otimes \partial_{\theta_1}\partial_{\theta_1} \otimes \partial_{\theta_{N+1}}\partial_{\theta_{N+1}}).$$

Now, let $A_{km} \in \mathbb{C}[\mathbf{x}]$ be such polynomials, that H_f is the following sum:

$$\mathrm{H}_f(x, g(x), x) = \sum_{\substack{(i,j)\neq(1,1)\\ (i,j)\neq(N+1,N+1)}} A_{ij}(x)\theta_i \otimes \theta_j + A_{11}(x)\theta_1 \otimes \theta_1 + A_{N+1,N+1}(x)\theta_{N+1} \otimes \theta_{N+1}$$

Then

$$[\mathrm{H}_f]^2 = 2[A_{11}A_{N+1,N+1}]\theta_1\theta_{N+1} \otimes \theta_1\theta_{N+1} + \sum_{(i,j,k,l)\neq(1,N+1,1,N+1)} [\tilde{A}_{i,j,k,l}]\theta_i\theta_j \otimes \theta_k\theta_l,$$

where $\tilde{A}_{i,j,k,l} \in \mathbb{C}[\mathbf{x}]$ are polynomials obtained by multiplication in $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\theta] \otimes \mathbb{C}[\theta]$. Consequently, we conclude that

$$\sigma_{g,g^{-1}} = -[A_{11}A_{N+1,N+1}],$$

and, finally, after precise calculations we obtain

$$[A_{N+1,N+1}] = [1]$$

$$A_{11} = [a_1 x_1^{2(a_1-1)} x_2 + \sum_{J_k\in A_R \wedge 1\in J_k} \varepsilon_{J_k} b_{k_1} x_1^{2(b_{k_1}-1)} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}}],$$

which implies the following relation in $\mathrm{Jac}(f, G)$:

$$(\xi_g)^2 = -[a_1 x_1^{2(a_1-1)} x_2 + \sum_{J_k\in A_R \wedge 1\in J_k} \varepsilon_{J_k} b_{k_1} x_1^{2(b_{k_1}-1)} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}}]\xi_{id}.$$

Lastly, by switching to $\mathrm{Jac}(\bar{f})$ we may write the relation achieved by taking the partial derivative:

$$[\frac{\partial f}{\partial x_1}] = [x_{N+1}^2 + a_1 x_1^{a_1-1} x_2 + \sum_{J_k\in A_R \wedge 1\in J_k} \varepsilon_{J_k} b_{k_1} x_1^{b_{k_1}-1} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}}] = [0]$$

$$[x_{N+1}]^2 = -[a_1 x_1^{a_1-1} x_2 + \sum_{J_k\in A_R \wedge 1\in J_k} \varepsilon_{J_k} b_{k_1} x_1^{b_{k_1}-1} x_{k_2}^{b_{k_2}} \dots x_{k_l}^{b_{k_l}}].$$

Therefore, the original claim is established. □

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Anton Rarovskii, Faculty of Mathematics, National Research University Higher School of Economics, Usacheva str., 6, 119048 Moscow, Russian Federation, and
Skolkovo Institute of Science and Technology, Nobelya str., 3, 121205 Moscow, Russian Federation

Email address: aararovskiy@edu.hse.ru