UNIVERSALLY EQUIDIMENSIONAL MORPHISMS AND WEAKLY-1-RATIONAL OR WEAKLY-2-RATIONAL SINGULARITIES

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ABSTRACT. This article aims to study the behavior of certain types of singularities in a universally equidimensional morphism (i.e., open with constant pure-dimensional fibers). These singularities are those of reduced complex spaces of pure dimension m for which the sheaf $\mathcal{L}_Z^m$ (whose sections are meromorphic forms that extend analytically over any desingularization of Z) has depth at least two and are called *weakly-1-rational* and denoted $\mathfrak{WR}^1$; for spaces whose singular locus is of codimension at least two, they are called *weakly-2-rational* and denoted $\mathfrak{WR}^2$. Our study focuses on the possibility of transferring this type of singularities from the total space to the base, from the base and the fibers to the total space, and from the latter to the fibers.

The present work aims to study certain types of singularities in families of subspaces described by universally equidimensional morphisms (i.e. open with constant fiber dimension) using the sheaves $\omega_Z^\bullet$ and $\mathcal{L}_Z^\bullet$ studied in [K3] and [K4]. Recall that in the top degree, $\mathcal{L}_Z^m$ is the sheaf of meromorphic forms whose pullback extends analytically over any desingularization, and ω_Z^m is Grothendieck's dualizing sheaf.

It has been known since R. Elkik's article [E78] that any deformation of a rational singularity over a base with rational singularities remains rational; stated in the algebraic framework (schemes of finite type over a field of characteristic zero), it remains valid in the complex analytic setting. Since the problem is local on each fiber and on the total space, the rationality of fiber singularities requires both the morphism to be flat and the source to be Cohen-Macaulay. In this construction, flatness plays a crucial role as it allows, on one hand, to lift any non-zero divisor to a non-zero divisor at the level of local rings, and on the other hand, to reveal a non-trivial relationship between the absolute and relative dualizing complexes. This leads us to primarily consider the case of a family of divisors parameterized by a smooth base, and mainly by a smooth curve. The general construction is achieved by induction on the base dimension and by applying Hironaka's desingularization theorem [H] to reach the case of a singular base.

As in the complex analytic setting, it is rare for a geometric situation without special conditions to exhibit Cohen-Macaulay structures, so it seemed natural to try to free ourselves from this constraint while maintaining a certain type of singularity that is, in some sense, close to rational ones. It is in this spirit that da Silva and Andreatta study in [AS84] the deformation of normal spaces whose sheaves $\omega_Z^\bullet$ and $\mathcal{L}_Z^\bullet$ coincide in maximal degree which they call *weakly rational* singularities. More precisely, a normal complex space X of pure dimension m has a weakly rational singularity at a point x if, for a desingularization $\varphi : \tilde{X} \to X$, we have the vanishing $(\mathbb{R}^{m-1}\varphi_* O_{\tilde{X}})_x = 0$. They show, following the method developed in [E78], that any deformation of a weakly rational singularity over a *smooth* base remains weakly rational.

This generalization remains partial as it confines them to the case of a smooth base without hope of addressing the singular case through Hironaka's desingularization theorem [H] (the arguments developed in [E78] are unsuitable for the weakly rational case).

In the following work and unless explicitly stated otherwise, we only consider reduced complex spaces of pure dimension without any normality condition. If the space is not normal or more generally if its singular locus is of codimension not greater than two, the cohomological vanishing condition above

obviously does not imply the equality of the sheaves $\omega_Z^\bullet$ and $\mathcal{L}_Z^\bullet$ in maximal degree and consequently, these are two distinct conditions. However, this is the case if, in addition to this condition, we have an identification of these two sheaves outside codimension at least two. These remarks lead us to denote by $\mathfrak{WR}^1$ the class of singularities of *weakly-1-rational* type (satisfying only the cohomological condition) and by $\mathfrak{WR}^2$ the class that we will call *weakly-2- rational*, that is, the class of reduced spaces of type $\mathfrak{WR}^1$ with a singular locus of codimension at least two (note that the latter corresponds to the weakly rationality of [AS84] if the space is normal). We provide some properties and characterizations of these classes in the preliminary section (for example, $\mathfrak{WR}^1$ corresponds precisely to reduced complex spaces Z of pure dimension m on which the sheaf $\mathcal{L}_Z^m$ has depth at least two).

The main objective of this work is to generalize [AS84] in three possible directions, namely:

• weakening the flatness condition by considering universally equidimensional morphisms (open morphisms with constant fiber dimension),

• considering singularities of type $\mathfrak{WR}^1$ or $\mathfrak{WR}^2$ (cf. Definition 1.2 and Definition 1.3),

• allowing singular bases with these types of singularities.

Leading to one of the main results stating that:

If the fibers of a universally equidimensional morphism have singularities of type $\mathfrak{WR}^2$ (resp. $\mathfrak{WR}^1$) over a base with singularities of type $\mathfrak{WR}^2$ (resp. $\mathfrak{WR}^1$), then the total space (or source) is also of type $\mathfrak{WR}^2$ (resp. $\mathfrak{WR}^1$).

More precisely, we show:

Theorem 0.1. *Stability under direct image for the classes $\mathfrak{R}$, $\mathfrak{WR}^1$ and $\mathfrak{WR}^2$:*

Let $\pi : Z \to S$ be a n-geometrically flat morphism of reduced complex spaces of pure dimensions m and r respectively. If Z has singularities of rational, weakly-1-rational, or weakly -2-rational type, then S has singularities of the same type.

Theorem 0.2. *Universally equidimensional deformation of the class $\mathfrak{WR}^2$: Let $\pi : Z \to S$ be a universally-n-equidimensional morphism of reduced complex spaces of pure dimension m and r respectively. If S and the fibers are of type $\mathfrak{WR}^2$ then so is Z.*

Theorem 0.3. *Universally equidimensional and reduced deformation of the class $\mathfrak{WR}^1$:*

Let $\pi : Z \to S$ be a universally-n-equidimensional morphism of reduced complex spaces of pure dimension whose base and fibers are of type $\mathfrak{WR}^1$. Then, Z is of type $\mathfrak{WR}^1$, if one of the following two conditions holds:

(i) *Z is locally of pure dimension m and π has reduced fibers, or*

(ii) *if $\mathrm{Sing}(\pi)$, the singular locus of π, satisfies the incidence condition:*

$$(P) \quad \overline{\overset{\circ}{\pi^{-1}(s) \cap \mathrm{Sing}(\pi)}} = \emptyset \ \text{in} \ \pi^{-1}(s), \ \forall \, s \in S$$

Theorem 0.4. *A weak converse:*

Let $\pi : Z \to S$ be a universally-n-equidimensional morphism of reduced complex spaces of pure dimension with S of type $\mathfrak{WR}^2$ (resp. $\mathfrak{WR}^1$) admitting a simultaneous resolution. If Z is of type $\mathfrak{WR}^2$ (resp. $\mathfrak{WR}^1$) then so is π.

In what follows, most often, our complex spaces are reduced and of pure finite-dimensional unless expressly mentioned otherwise.

1. NOTATION AND DEFINITIONS

Definition 1.1. *A reduced complex analytic space Z of dimension m has a* rational singularity *at a point z if for a desingularization $\varphi : \tilde{Z} \to Z$, we have $\mathbb{R}\varphi_* O_{\tilde{Z}} \simeq O_Z$ (i.e., $(\mathbb{R}^j \varphi_* O_{\tilde{Z}})_z = 0$ for all $j > 0$).*

Definition 1.2. *We will say that a reduced complex analytic space Z has* semi-rational singularity *if his normalization has rational singularities.*

Definition 1.3. *A complex analytic space Z of dimension m has a* weakly-1-rational singularity *at z if for a desingularization $\varphi : \tilde{Z} \to Z$, we have $(\mathbb{R}^{m-1}\varphi_* O_{\tilde{Z}})_z = 0$.*

Definition 1.4. *A complex analytic space Z of dimension m has a* weakly-2-rational singularity *at z if the canonical injective morphism $\mathcal{L}_{Z,z}^m \to \omega_{Z,z}^m$ is surjective.*

In the category of reduced complex analytic spaces, we will denote by $\mathfrak{R}$ (resp. $\mathfrak{WR}^1$, resp. $\mathfrak{WR}^2$) the class of rational singularities (resp. weakly-1- rational, resp. weakly-2-rational) and we will say that a complex space Z is of type $\mathfrak{R}$ (resp. $\mathfrak{WR}^1$, resp. $\mathfrak{WR}^2$) if it has rational singularities (resp. weakly-1-rational, resp. weakly-2-rational).

Definition 1.5. *We will say that a complex space is of type k-$\mathfrak{WR}^1$ if the sheaves $\mathcal{L}_Z^j$ have depth at least two for all integers $j \leq k$. It will be of type $k - \mathfrak{WR}^2$ if, for all $j \leq k$, the injective morphism $\mathcal{L}_Z^j \to \omega_Z^j$ is bijective.*

Definition 1.6. *We will say that a morphism $\pi : Z \to S$ of reduced complex spaces is* reduced[1] *(resp.* normal*) or of type $\mathfrak{WR}^1$ (resp. $\mathfrak{WR}^2$) if S and its fibers are reduced (resp. normal) or of type $\mathfrak{WR}^1$ (resp. $\mathfrak{WR}^2$).*

We say it is generically reduced or that the generic fiber is reduced if it is reduced over a dense subset of S.

Notation 1.7. In the sequel, we write shortly Z is CM for Z is a Cohen-Macaulay complex space and $\mathcal{F}$ is CM (resp. MCM) for $\mathcal{F}$ is a Cohen-Macaulay(resp. maximal Cohen-Macaulay) coherent sheaf.

Definition 1.8. *Let $n \in \mathbb{N}$. We will say that a morphism $\pi : Z \to S$ of complex spaces is* universally-n-equidimensional *if it is open with constant fiber dimension n. It is geometrically flat if these fibers can be equipped with suitable multiplicities to form an analytic family of n-cycles (cf. [BM], [BM2], [K1], [K2]).*

Geometric flatness imposes the following property on the morphism:

For every point z of Z (naturally in the fiber $\pi^{-1}(\pi(z))$) and every local factorization at z (relative to the fiber $\pi^{-1}(\pi(z))$),

$$\begin{array}{ccc} Z & \xrightarrow{\;f\;} & S \times U \\ & {}_{\pi}\searrow & \downarrow{}^{q} \\ & & S \end{array}$$

where f is a finite, surjective, and open morphism, q is the canonical projection, and U is a relatively compact Stein open subset of some numerical space $\mathbb{C}^n$, there exists a trace morphism $f_* O_Z \to O_{S \times U}$ inducing a morphism (also of trace type) $f_* f^* \Omega_{S \times U/S}^n \to \Omega_{S \times U/S}^n$ that extends naturally to a trace morphism

$$f_* \Omega_{Z/S}^n \to \Omega_{S \times U/S}^n.$$

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[1] In standard terminology, π is said *reduced* (resp. *normal*) at z, if π is flat at z and $\pi^{-1}(\pi(z))$ is reduced (resp. normal) at z. The non-reduced (resp. non-normal) locus $\mathrm{NRed}(\pi) := \{z \in Z : \pi \text{ not reduced at } z\}$ (resp. $\mathrm{NN}(\pi) := \{z \in Z : \pi \text{ not normal at } z\}$ is analytically closed in Z (cf.[Fi], **Prop.3.22**)

As this notion is not familiar to non-specialists, we gather a few properties and will give explicit examples in the final remark (Remark 8).

(1) Some properties of geometrically flat maps:

(i) Every pure equidimensional morphism of reduced complex spaces with normal base is geometrically flat.

(ii) Surjective flat morphism and finite Tor-dimension morphism between reduced complex spaces are geometrically flat map (cf [Si], **II**; [K2], **cor.5, p.35**). The converse is false as we can see with the simple examples of the parametrization of the cone or the Whitney umbrella in $\mathbb{C}^3$ given by $(u,v) \to (u^2, v^2, uv)$ and $(u,v) \to (u^2, v, uv)$, respectively.

(iii) Unlike flatness, the geometric flatness of a morphism $\pi : X \to S$ depends only on the reduction $\mathrm{Red}(X)$. The Douady example (cf [Fi],p.151) of the finite projection of

$$X := \{(x,a,b) \in \mathbb{C}^3 : x^3 + ax + b = 4a^3 + 27b^2 = 0\}$$

over $S := \{(a,b) \in \mathbb{C}^2 : 4a^3 + 27b^2 = 0\}$ is flat map which is not flat on

$$\mathrm{Red}(X) := \{(x,a,b) : x^3 + ax + b = 6ax^2 - 9bx + 4a^2 = 0\},$$

but geometrically flat on it. As we see, this notion depend on the reduced structure of X by construction of the analytic cycle space of Barlet which different from Douady space.

(iv) ([BM], **Prop.A1,p.839**). Let $\pi : X \to S$ be an equidimensional and open morphism of reduced complex spaces. Then, the set of points of X at which π is not geometrically flat is analytically closed.

(v) ([Si], **II, Prop.5.8**, [BM2]): Let $\pi : X \to S$ be an open and surjective morphism of complex spaces whose generic fibers are of constant dimension over a weakly normal S. Then π is geometrically flat if and only if for every point $x \in X$ and a certain projection (hence for every) $\eta : W \to U$, $x \in W$ open in X and $V = \pi(W)$, the ramified covering $(\pi, \eta) : W \to V \times U$ is of constant degree.

(vi) ([Si],**II**): Let $\pi : X \to S$ be an open and surjective morphism from a complex space X locally of pure dimension onto a locally irreducible complex space S. Then π is geometrically flat.

(vii) ([K1], **Prop7,p.50**): If $\pi : X \to S$ is a geometrically flat morphism with X weakly normal (resp. normal) then S is weakly normal (resp. normal).

(viii) If $\pi : Z \to S$ is geometrically flat with normal fibers and normal base then Z is normal.

It is an immediate consequence of the fact that open morphism of complex spaces with reduced base and reduced fibres is flat[2] which is deduced from:

(a) a morphism $\pi : X \to M$ with reduced fibers over a smooth base is flat if and only if it is open ([Kan], [OKY], [GR-Ke], [Fi] **p.157**),

(b) a morphism of complex spaces $\pi : X \to Y$ with Y reduced is flat if and only if for each point y of Y and each holomorphic map $\varphi : \mathbb{D} \to Y$ such that $\varphi(0) = y$ and $\mathbb{D} := \{t \in \mathbb{C} : |t| < 1\}$, the induced morphism (by base change) $\pi_\varphi : X_\mathbb{D} \to \mathbb{D}$ is flat (see, for example, [Fu], **Appendix p.208**),

(c) the property of being reduced and of constant dimension for the fibers, open for a morphism are invariant under arbitrary base change.

(2) Analogy with flat morphisms:

Geometrically flat morphisms possess some properties analogous to those of flat morphisms and of finite Tor-dimension between reduced complex spaces, namely:

(ii) they are stable under open restrictions on X and S. The property of being analytically geometrically flat is an open condition on X.

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[2]This is a known result in algebraic geometry (cf [E.G.A.4], **Cor.15.2.3**)

(ii) they are stable under base changes. Note that morphisms of finite Tor-dimension satisfy this property only for cohomologically transverse base changes.

(iii) As universally equidimensional morphisms, they always factorize, at least locally with respect to each of their fibers, into finite morphisms followed by projections. But being geometrically flat (essentially by the construction of the cycle space), the finite morphisms of the local factorization are always geometrically flat. Let us point out that the local factorization of a finite Tor-dimension morphism already possesses this hereditary property since the finite morphism of the factorization is of finite Tor-dimension but generally false in the flat case because otherwise the fibers would be Cohen-Macaulay spaces!

(iv) A major difference between flatness and geometric flatness is the non-stability under composition (outside of cases where the bases are normal or weakly normal).

2. Some properties of $\mathfrak{R}$, $\mathfrak{WR}^1$ and $\mathfrak{WR}^2$ singularities

2.1. **Singularities of $\mathfrak{R}$-type.** One cannot cite all the existing literature on rational singularities (see Mumford, Artin, Steenbrink-van Straten, Burns, Laufer, Wahl, Lipman, Kollar, Kovacs, Schwede and many other authors), but it is impossible not to recall the now-classic result known as the *Kempf rationality criterion* [Ke], valid in all characteristics:

Proposition 2.1. *: Let Z be a reduced complex space. Then, the following equivalences hold:*
(i) *Z is of type $\mathfrak{R}$*

(ii) *The canonical morphism $O_Z \to \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])$ is a quasi-isomorphism.*

(iii) *Z is of type $\mathfrak{WR}^2$ and CM.*

(iv) *The sheaf $\mathcal{L}_Z^m$ is MCM and Z is normal.*

Proof. The equivalence between **(i)** and **(iii)** is precisely Kempf's criterion.

• **(i)**$\Longleftrightarrow$**(ii)**: This is a straightforward application of analytic duality for a proper morphism and the Grauert-Riemenschneider vanishing theorem. Indeed, if $\varphi : \tilde{Z} \to Z$ is a given desingularization, the relative analytic duality for a proper morphism from [RRV71] yields the isomorphism

$$\mathbb{R}\varphi_* O_{\tilde{Z}} \simeq \mathbb{R}\varphi_* \mathbb{R}\mathcal{H}om(\Omega_{\tilde{Z}}^m, \Omega_{\tilde{Z}}^m) \simeq \mathbb{R}\mathcal{H}om(\mathbb{R}\varphi_* \Omega_{\tilde{Z}}^m[m], \mathcal{D}_Z^\bullet)$$

However, the Grauert-Riemenschneider vanishing theorem [GR70] (see [Ta] for the analytic case) gives the quasi-isomorphism $\mathbb{R}\varphi_* \Omega_{\tilde{Z}}^m = \mathcal{L}_Z^m$ and, consequently,

$$\mathbb{R}\varphi_* O_{\tilde{Z}} \simeq \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])$$

(ii)$\Longleftrightarrow$**(iii)**:

• For the implication, note that by biduality, condition **(ii)** is equivalent to the quasi-isomorphism $\mathcal{L}_Z^m \simeq \mathcal{D}_Z^\bullet[-m]$. Consequently, the dualizing complex reduces to a single sheaf in degree $-m$. Now, by construction, $\omega_Z^m = \mathcal{H}^{-m}(\mathcal{D}_Z^\bullet)$ and thus $\mathcal{L}_Z^m = \omega_Z^m$. Therefore, Z is CM and of type $\mathfrak{WR}^2$.
To ensure the normality of Z, observe that the canonical morphism

$$\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m) \to \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])$$

induces, at the level of degree 0 cohomology, the morphism $\mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m) \to O_Z$ which is necessarily injective since it is generically bijective between torsion-free sheaves. Since there is already a natural morphism $O_Z \to \mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m)$ that is also injective for the same reasons, we deduce the isomorphism

$$O_Z \simeq \mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m)$$

From this, considering the natural injections

$$O_Z \subset \mathcal{L}_Z^0 \subset \mathcal{H}om(\mathcal{L}_Z^m, \mathcal{L}_Z^m) \subset \mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m)$$

the isomorphism $O_Z \simeq \mathcal{L}_Z^0$ characterizes the normality of Z.

• The converse is evident since, Z being Cohen-Macaulay, we have

$$O_Z \simeq \mathbb{R}\mathcal{H}om(\omega_Z^m[m], \omega_Z^m[m]) \simeq \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m]).$$

(iii)⟺(iv):

• The quasi-isomorphism $O_Z \simeq \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])$ precisely means that

$$\mathcal{H}^j(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m]) = 0 \,\forall j \neq 0 \text{ and } \mathcal{H}^0(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m]) \simeq O_Z.$$

As seen above, the second condition implies the normality of Z. As for the first, it ensures that for every Stein open subset U of Z, by the duality of [RR70] or [AK], we have

$$\mathrm{Ext}^j(U; \mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m]) \simeq \left(\mathrm{H}_c^{m-j}(U, \mathcal{L}_Z^m)\right)'$$

However, the depth characterization from ([Ba1], ([BS],**I, §3, Theorem 3.6,p.77**)) guarantees that the vanishing conditions

$$\mathrm{H}_c^k(U, \mathcal{L}_Z^m) = 0, \,\forall k \neq m$$

are equivalent to $\mathcal{L}_Z^m$ being MCM.

• Conversely, if $\mathcal{L}_Z^m$ is MCM, then by the duality mentioned earlier, we have the vanishing conditions $\mathcal{H}^j(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])) = 0 \,\forall j \neq 0$, and thus

$$\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m]) \simeq \mathcal{H}^0(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])).$$

It only remains to prove that $\mathcal{H}^0(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m]) \simeq O_Z$.

Since

$$\mathcal{H}^j(\mathcal{L}_Z^m) = \mathcal{H}^{-j}(\mathcal{D}_Z^\bullet[-m]) = 0, \,\forall j > 0$$

there is a canonical morphism

$$\mathcal{H}^0(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])) \rightarrow \mathcal{H}om(\mathcal{H}^0(\mathcal{L}_Z^m), \mathcal{H}^0(\mathcal{D}_Z^\bullet[-m])) = \mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m)$$

noting that there is an equally natural arrow in the other direction, as mentioned in the proof of the equivalence between **(ii)** and **(iii)**. The arguments presented in that proof show that we have an isomorphism

$$\mathcal{H}^0(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])) \simeq \mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m).$$

But Z being normal, the natural morphism $O_Z \rightarrow \mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m)$ is necessarily bijective since it is so on the regular part and the sheaves have depth at least two. Hence, the isomorphism $\mathcal{H}^0(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])) \simeq O_Z$ and, consequently, the quasi-isomorphism

$$O_Z \simeq \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m]).$$

$\square$

Note in passing that we have implicitly established:

Corollary 2.2. *With the notations and hypotheses of Proposition 2.1, the following assertions are equivalent:*

(i) *Z has semi-rational singularities,*
(ii) *the canonical morphism $\mathcal{L}_Z^0 \rightarrow \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])$ is a quasi-isomorphism,*
(iii) *the normalization of Z has rational singularities,*
(iv) *$\mathcal{L}_Z^m$ is a MCM sheaf.*

Proof. **(i)⟺(ii)**: This is essentially due to the definition. Indeed, if $\varphi : \widetilde{Z} \rightarrow Z$ is a given desingularization of Z, the semi-rationality of the singularities of Z is equivalent to the vanishing $\mathbb{R}^j\varphi_*O_{\widetilde{Z}} = 0$ for all non-zero integers j and, therefore, to the vanishing of cohomology $\mathcal{H}^j(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])) = 0$ for all non-zero j since $\mathbb{R}\varphi_*O_{\widetilde{Z}} \simeq \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet[-m])$, where the degree 0 cohomology corresponds precisely to the sheaf $\mathcal{L}_Z^0$.

(i)$\Longleftrightarrow$(iii): It suffices to consider a commutative diagram of the form

$$
\begin{array}{ccc}
 & \widetilde{Z} & \\
{}^{\tilde{\varphi}}\swarrow & & \searrow{}^{\varphi} \\
\overline{Z} & \xrightarrow{\ \ \nu\ \ } & Z
\end{array}
$$

where ν is the normalization of Z, and φ is a desingularization dominating this normalization. Then, by the finiteness of ν, we have

$$\mathbb{R}^j \varphi_* O_{\widetilde{Z}} = 0, \forall\, j \neq 0 \iff \mathbb{R}^j \tilde{\varphi}_* O_{\widetilde{Z}} = 0, \forall\, j \neq 0$$

using the fact that for any finite morphism $f : X \to Y$ and any coherent sheaf $\mathcal{F}$ on X, $f_* \mathcal{F} = 0 \implies \mathcal{F} = 0$ due to the canonical surjection $f^* f_* \mathcal{F} \to \mathcal{F}$.

Since $\overline{Z}$ is normal, $\tilde{\varphi}_* O_{\widetilde{Z}} = O_{\overline{Z}}$ and, thus, $\mathbb{R}\tilde{\varphi}_* O_{\widetilde{Z}} \simeq O_{\overline{Z}}$.

To show that **(iv)** is equivalent to any of the other points, we use classical duality theorems. Since our assertions are local in nature, we may assume Z is Stein. Consider the Leray spectral sequences

$$\mathrm{E}_2^{i,j} := \mathrm{H}^i(Z, \mathbb{R}^j \varphi_* O_{\widetilde{Z}}) \Longrightarrow \mathrm{H}^{i+j}(\widetilde{Z}, O_{\widetilde{Z}})$$

$$\mathrm{E'}_2^{i,j} := \mathrm{H}^i_c(Z, \mathbb{R}^j \varphi_* \Omega^m_{\widetilde{Z}}) \Longrightarrow \mathrm{H}^{i+j}_c(\widetilde{Z}, \Omega^m_{\widetilde{Z}})$$

which, as is known, degenerate by virtue of Grauert's coherence theorem for higher direct images under a proper morphism ([GR0]) and Cartan's Theorem B ([C]), for the first, and by [GR70] for the second.

Then, the topological duality isomorphisms ([RR70], [Se])

$$\mathrm{Ext}^j(Z; \mathcal{L}^m_Z, \mathcal{D}^\bullet_Z[-m]) \simeq \big(\mathrm{H}^{m-j}_c(Z, \mathcal{L}^m_Z)\big)'$$

$$\mathrm{H}^j(\widetilde{Z}, O_{\widetilde{Z}}) \simeq \big(\mathrm{H}^{m-j}_c(\widetilde{Z}, \Omega^m_{\widetilde{Z}})\big)'$$

combined with the topological isomorphisms given by the degeneration of the spectral sequences

$$\mathrm{H}^0(Z, \mathbb{R}^j \varphi_* O_{\widetilde{Z}}) \simeq \mathrm{H}^j(\widetilde{Z}, O_{\widetilde{Z}}), \ \ \mathrm{H}^i_c(Z, \mathcal{L}^m_Z) \simeq \mathrm{H}^i_c(\widetilde{Z}, \Omega^m_{\widetilde{Z}})$$

show that, for all non-zero integers j,

$$\mathrm{H}^{m-j}_c(Z, \mathcal{L}^m_Z) = 0 \iff \mathcal{H}^j(\mathbb{R}\mathcal{H}om(\mathcal{L}^m_Z, \mathcal{D}^\bullet_Z[-m])) = 0 \iff \mathrm{H}^j(\widetilde{Z}, O_{\widetilde{Z}}) = 0 \iff \mathbb{R}^j \varphi_* O_{\widetilde{Z}} = 0.$$

$\square$

A more or less independent result is given by the

Corollary 2.3. *Let Z be a complex space. Then, we have*

(i) Z *is normal if and only if* $\mathrm{Codim}(\mathrm{Sing}(Z)) \geq 2$ *and* $O_Z \simeq \mathcal{H}om(\omega^m_Z, \omega^m_Z)$.

(ii) Z *is CM if and only if* ω^m_Z *is MCM and* $O_Z \simeq \mathcal{H}om(\omega^m_Z, \omega^m_Z)$.

Proof. **(i)** The implication is obvious. For the converse, the given isomorphism shows that O_Z has depth at least two and, since by hypothesis $\mathrm{Sing}(Z)$ has codimension at least two,

$$\mathcal{H}^1_{\mathrm{Sing}(Z)}(O_Z) = \mathcal{H}^0_{\mathrm{Sing}(Z)}(O_Z) = 0,$$

which characterizes normality.

(ii) As with the sheaf $\mathcal{L}^m_Z$, this maximality property translates into cohomological vanishing and leads, with the same arguments, to the vanishing $\mathcal{H}^j(\mathbb{R}\mathcal{H}om(\omega^m_Z, \mathcal{D}^\bullet_Z[-m])) = 0$ for all $j \neq 0$. Hence the equivalence

$$\omega^m_Z \text{ is MCM} \iff \mathcal{H}om(\omega^m_Z, \omega^m_Z) \simeq \mathbb{R}\mathcal{H}om(\omega^m_Z, \mathcal{D}^\bullet_Z[-m]).$$

If we have $O_Z \simeq \mathbb{R}\mathcal{H}om(\omega^m_Z, \mathcal{D}^\bullet_Z[-m])$, we have necessarily by biduality $\mathcal{D}^\bullet_Z[-m] \simeq \omega^m_Z$ which means that Z is CM.

Conversely, we know that Z is CM if and only if $\mathcal{D}_Z^\bullet[-m] \simeq \omega_Z^m$ which implies that

$$\mathbb{R}\mathcal{H}om(\omega_Z^m, \mathcal{D}_Z^\bullet[-m])) \simeq O_Z$$

and then ω_Z^m is MCM. $\qquad\square$

2.1.1. **Quasi-smooth spaces and rational singularities.** A simple example of rational singularity is given by quotient singularities (cf. Mumford, Steenbrink-Straten, Burns). We can also mention that a large class of rational singularities is given by quasi-smooth spaces:

Definition 2.4. *([BM1]): A normal and reduced complex analytic space is said to be* quasi-smooth *if, for each point z of Z, there exists an open neighborhood U such that there exists a smooth variety V and a finite surjective morphism $f : V \to U$.*

Proposition 2.5. *Every quasi-smooth complex space Z of dimension m has rational singularities.*

Proof. We will use Kempf's criterion recalled in Proposition 2.1 by showing that Z is necessarily CM and that $\mathcal{L}_Z^m$ is isomorphic to the dualizing sheaf ω_Z^m.

To avoid overloading the text, we retain the notations with Z and V, keeping in mind that local models must be considered.

- **Z is CM:**

Since the morphism $f : V \to Z$ is a ramified covering of some degree k (because it is finite over a normal base), it induces a trace morphism $\mathcal{T}r : f_*O_V \to O_Z$ defined by $\mathcal{T}r(h)(t) := \sum_{x_j \in f^{-1}(t)} h(x_j)$ for any holomorphic function h on V. As previously stated, the pullback $O_Z \to f_*O_V$ followed by this trace identifies with the morphism $k.\mathrm{Id}$, and the sheaf O_Z is a direct summand of the sheaf f_*O_V. We deduce that the vanishing

$$\mathrm{H}_c^i(V, O_V) = \mathrm{H}_c^i(Z, f_*O_V) = 0, \ \forall i \neq m$$

implies the vanishing

$$\mathrm{H}_c^i(Z, O_Z) = 0, \ \forall i \neq m$$

which means that O_Z is maximal Cohen-Macaulay, i.e., Z is CM.

- **Isomorphism between $\mathcal{L}_Z^m$ and ω_Z^m:**

The sheaf O_Z being a direct summand of f_*O_V, $\mathcal{H}om(O_Z, \omega_Z^m)$ is also a direct summand of $\mathcal{H}om(f_*O_V, \omega_Z^m)$, meaning ω_Z^m is a direct summand of $f_*\Omega_V^m$. Hence, there is a nontrivial morphism from ω_Z^m to $f_*\Omega_V^m$. However, this is possible if and only if $\mathcal{L}_Z^m = \omega_Z^m$.

Indeed, consider the desingularization morphism $v : \widetilde{Z} \to Z$ and the resulting commutative base change diagram:

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\tilde{v}} & V \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Z} & \xrightarrow{v} & Z \end{array}$$

where $\widetilde{V}$ denotes the fiber product $V \times_Z Z$.

Then, applying the pullback $\tilde{v}^*$ to the morphism $f^*\omega_Z^m \to \Omega_V^m$ and using the commutativity of the diagram, we obtain the morphism $\tilde{f}^*v^*\omega_Z^m \to \Omega_{\widetilde{V}}^m$, or equivalently, $v^*\omega_Z^m \to \tilde{f}_*\Omega_{\widetilde{V}}^m$. Since $\tilde{f}$ is also a ramified covering, it induces a trace morphism $\tilde{f}_*\Omega_{\widetilde{V}}^m \to \Omega_{\widetilde{Z}}^m$. Thus, we have the arrow $v^*\omega_Z^m \to \Omega_{\widetilde{Z}}^m$, which gives rise to a morphism $\omega_Z^m \to \mathcal{L}_Z^m$. This morphism is necessarily injective because these two torsion-free sheaves coincide generically on Z. The conclusion follows since the sheaf $\mathcal{L}_Z^m$ naturally injects into the sheaf ω_Z^m. $\qquad\square$

Remark 2.6. **(i)** To show that Z is CM, one can prove that for any local parametrization $g : Z \to W$ (around a fixed point z in Z), the sheaf g_*O_Z is locally free. After sufficient localization, we obtain a commutative factorization diagram:

$$
\begin{array}{ccc}
V & \xrightarrow{\quad f \quad} & Z \\
 & \searrow_{h} \quad \swarrow_{g} & \\
 & W &
\end{array}
$$

The morphism h is a ramified covering and thus open. But since V and W are smooth, h is necessarily flat, and consequently, h_*O_V is locally free. Since O_Z is a direct summand of f_*O_V, g_*O_Z is then a direct summand of the locally free sheaf h_*O_V. Hence, g_*O_Z is locally free.

(ii) Another way to see this would be to consider relative duality for a proper morphism to show that the dualizing complex of Z reduces to a single sheaf placed in degree $-m$.

Indeed, we have $f^!\mathcal{D}_Z^\bullet = \mathcal{D}_V^\bullet = \Omega_V^m[m]$, and thus $\mathcal{H}^j(f^!\mathcal{D}_Z^\bullet[-m]) = 0$ for all $j \neq 0$. Since f is finite, we have (by construction) $f_*(f^!\mathcal{D}_Z^\bullet[-m]) = \mathbf{R}\mathcal{H}om(f_*O_V, \mathcal{D}_Z^\bullet[-m])$. Since we have a composition $O_Z \to f_*O_V \to O_Z$ identified with the identity morphism, the sheaf O_Z is a direct summand of f_*O_V. It follows that the cohomology of the complex $\mathcal{D}_Z^\bullet[-m]$ is also a direct summand of the cohomology of the complex $\mathbf{R}\mathcal{H}om(f_*O_V, \mathcal{D}_Z^\bullet[-m])$, i.e., of the complex $f_*(f^!\mathcal{D}_Z^\bullet[-m])$. However, the vanishing of the cohomology of $f^!\mathcal{D}_Z^\bullet[-m])$ in all nonzero degrees, combined with the finiteness of f, leads to the desired conclusion since the dualizing complex of Z has cohomology only in degree $-m$.

Corollary 2.7. *Let Z be a quasi-smooth complex space. Then:*
(i) *For every integer $k \geq 0$, the O_Z-coherent sheaves ω_Z^k and $\mathcal{L}_Z^k$ coincide and are MCM.*
(ii) *Moreover, the canonical morphism $\omega_Z^k \to \mathbf{R}\mathcal{H}om(\omega_Z^{m-k}, \omega_Z^m)$ is a quasi-isomorphism.*

Proof. **(i)** To show the equality of sheaves $\mathcal{L}_Z^k = \omega_Z^k$ for every integer k, it suffices to exhibit a nontrivial morphism from ω_Z^k to $\mathcal{L}_Z^k$. In this context, the existence of such a morphism is equivalent to ω_Z^k being a direct summand of the sheaf $f_*\Omega_V^k$.

To see this, first observe that the sheaf $\mathcal{L}_Z^k$ is a direct summand of $f_*\Omega_V^k$ since it is equipped with a pullback $\mathcal{L}_Z^k \to f_*\Omega_V^k$ and a trace morphism $f_*\Omega_V^k \to \mathcal{L}_Z^k$ whose composition yields the identity on $\mathcal{L}_Z^k$. It follows that for every integer k, the sheaves $\mathcal{H}om(\mathcal{L}_Z^{m-k}, \omega_Z^m)$ are direct summands of $\mathcal{H}om(f_*\Omega_V^{m-k}, \omega_Z^m)$. But thanks to the isomorphism $\omega_Z^k \simeq \mathcal{H}om(\mathcal{L}_Z^{m-k}, \omega_Z^m)$ due to the normality of Z, we see that ω_Z^k is also a direct summand of $f_*\Omega_V^k$, and we conclude as in Proposition 2.5 to deduce that $\mathcal{L}_Z^k \simeq \omega_Z^k$.

(ii) Now, let us show that for every integer k, the natural morphism $\omega_Z^k \to \mathbf{R}\mathcal{H}om(\omega_Z^{m-k}, \omega_Z^m)$ is in fact a quasi-isomorphism (which constrains the complex $\mathbf{R}\mathcal{H}om(\omega_Z^{m-k}, \omega_Z^m)$ to be acyclic).

Since $\Omega_V^k \simeq \mathcal{H}om(\Omega_V^{m-k}, \Omega_V^m) \simeq \mathbf{R}\mathcal{H}om(\Omega_V^{m-k}, \Omega_V^m)$, relative duality for the proper morphism f gives $f_*\Omega_V^k \simeq \mathbf{R}\mathcal{H}om(f_*\Omega_V^{m-k}, \omega_Z^m)$, and thus the vanishing of the cohomology sheaves $\mathcal{E}xt^j(f_*\Omega_V^{m-k}, \omega_Z^m)$ for all $j \geq 1$. Moreover, by **(i)**, the sheaf ω_Z^{m-k} (resp. $\mathcal{L}_Z^{m-k}$) is a direct summand of $f_*\Omega_V^{m-k}$, and consequently, the cohomology sheaf $\mathcal{E}xt^j(\omega_Z^{m-k}, \omega_Z^m)$ (resp. $\mathcal{E}xt^j(\mathcal{L}_Z^{m-k}, \omega_Z^m)$) is a direct summand of $\mathcal{E}xt^j(f_*\Omega_V^{m-k}, \omega_Z^m)$, which is known to vanish.

Since Z is CM and Stein, by duality, this is exactly equivalent to writing

$$
\mathrm{H}_c^j(Z, \omega_Z^k) = \mathrm{H}_c^j(Z, \mathcal{L}_Z^k) = 0, \ \forall\, k \geq 0, \ \forall\, j \neq m
$$

meaning that these sheaves are MCM. $\qquad\square$

Remark 2.8. Of course, **(i)** can be deduced from ([KeSc], *Theorem (1.2), Theorem (1.3)*), which we recall a little later, but this is not necessary.

2.1.2. **Singularities of type $\mathfrak{WR}^1$, $\mathfrak{WR}^2$.**

Proposition 2.9. *Let Z be a complex space. Then, the following equivalences hold:*
 (i) Z *is of type $\mathfrak{WR}^1$,*
 (ii) $\mathcal{L}_Z^m$ *has depth at least two in Z,*
 (iii) *the normalization of Z is of type $\mathfrak{WR}^2$,*
 (iv) *for every integer j, the sheaves $\mathcal{L}_Z^j$ are ω_Z^m (resp. $\mathcal{L}_Z^m$, resp. $\mathcal{L}_Z^0$)-reflexive.*

Proof. Let $\nu : \overline{Z} \to Z$ denote the normalization of Z.

(i)$\Longleftrightarrow$(ii): Relative analytic duality for a proper morphism [RRV71] and the Grauert-Riemenschneider vanishing theorem [GR70] or [S] yield the quasi-isomorphisms

$$\mathbb{R}\mathcal{H}om(\mathbb{R}\varphi_*\Omega_{\tilde{Z}}^m, \mathcal{D}_Z^\bullet) = \mathbb{R}\varphi_*\mathbb{R}\mathcal{H}om(\Omega_{\tilde{Z}}^m, \Omega_{\tilde{Z}}^m[m]) = \mathbb{R}\varphi_*O_{\tilde{Z}}[m]$$

and consequently, the isomorphism of cohomology sheaves

$$\mathcal{E}xt^j(\mathcal{L}_Z^m, \mathcal{D}_Z^\bullet) \simeq \mathbb{R}^{m+j}\varphi_*O_{\tilde{Z}}$$

Considering strong duality (cf. [AK]) on these cohomology groups endowed with a separated locally convex topological vector space structure since $\mathrm{H}_c^j(U; \mathcal{L}_Z^m) = \mathrm{H}_c^j(\varphi^{-1}(U); \Omega_{\tilde{Z}}^m)$, we obtain, in particular, for every Stein open subset of Z, the isomorphism of separated topological vector spaces

$$\mathcal{E}xt^j(U; \mathcal{L}_Z^m, \mathcal{D}_Z^\bullet) \simeq (\mathrm{H}_c^{-j}(U, \mathcal{L}_Z^m))'$$

Now, according to ([Ba1], ([BS] **I, §3, Theorem 3.6,p.77**)), we have $\mathrm{depth}(\mathcal{L}_Z^m) \geq 2$ if and only if, for every Stein open subset U of Z, $\mathrm{H}_c^1(U, \mathcal{L}_Z^m) = \mathrm{H}_c^0(U, \mathcal{L}_Z^m) = 0$. But this is equivalent to the vanishing of the sheaves $\mathcal{E}xt^j(U; \mathcal{L}_Z^m, \mathcal{D}_Z^\bullet) = 0$ for $j = -1, 0$ and thus to the vanishing of the cohomology groups (Fréchet topological vector spaces) $\Gamma(U, \mathbb{R}^i\varphi_*O_{\tilde{Z}}) = 0$, for $i = m - 1$ and $i = m$. Since U is an arbitrary Stein open subset and Z can always be covered by a countable family of Stein open subsets, this proves the desired equivalence.

(ii)$\Longleftrightarrow$(iii): This is evident since $\mathcal{L}_Z^m = \nu_*\mathcal{L}_{\overline{Z}}^m$ and depth is preserved under finite direct image. Thus, the equality $\mathrm{depth}(\mathcal{L}_{\overline{Z}}^m) = \mathrm{depth}(\mathcal{L}_Z^m)$ shows that if Z is of type $\mathfrak{WR}^1$, the same holds for $\overline{Z}$. Moreover, the sheaves $\mathcal{L}_{\overline{Z}}^m$ and $\omega_{\overline{Z}}^m$, both of depth at least two, coincide on the regular part and hence everywhere since the singular locus has codimension at least two. Therefore, $\overline{Z}$ is of type $\mathfrak{WR}^2$.

Conversely, if $\overline{Z}$ is of type $\mathfrak{WR}^2$, we have $\mathcal{L}_{\overline{Z}}^m \simeq \omega_{\overline{Z}}^m$ and thus $\mathcal{L}_Z^m = \nu_*\mathcal{L}_{\overline{Z}}^m \simeq \nu_*\omega_{\overline{Z}}^m$. Hence, by the invariance of depth under finite direct image, $\mathrm{depth}(\mathcal{L}_Z^m) \geq 2$, meaning Z is of type $\mathfrak{WR}^1$.

(iii)$\Longleftrightarrow$(iv):
 • The implication **(iv)$\Longrightarrow$(iii)** is clear since for $j = m$, the ω_Z^m-reflexivity translates to the isomorphism

$$\mathcal{L}_Z^m \simeq \mathcal{H}om(\mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m), \omega_Z^m)$$

which implies $\mathrm{depth}(\mathcal{L}_Z^m) \geq 2$ since this holds for the sheaf ω_Z^m. As above, we also have $\mathrm{depth}(\mathcal{L}_{\overline{Z}}^m) \geq 2$ and thus $\overline{Z}$ is of type $\mathfrak{WR}^2$.

 • To establish the converse, note that for any finite modification $f : X \to Y$ of reduced complex spaces, the canonical morphism

$$f_*\mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(f_*\mathcal{F}, f_*\mathcal{G})$$

is an isomorphism if $\mathcal{G}$ is torsion-free. This is easily verified using the natural exact sequence

$$0 \to \mathcal{N} \to f^*f_*\mathcal{F} \to \mathcal{F} \to 0$$

to which we apply the functor $\mathcal{H}om(-, \mathcal{G})$ to obtain the exact sequence

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(f^*f_*\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{N}, \mathcal{G}).$$

Then, knowing that $\mathcal{N}$ is torsion and $\mathcal{G}$ is not, applying the functor f_* and the adjunction formula, we obtain the desired isomorphism.

On the other hand, it is known that for any finite morphism $f : X \to Y$ of pure m-dimensional complex spaces, we have

$$f_* \mathcal{H}om(\mathcal{F}, \omega_X^m) \simeq \mathcal{H}om(f_*\mathcal{F}, \omega_Y^m).$$

Thus, if $\overline{Z}$ is of type $\mathfrak{WR}^2$, the equality of the sheaves $\mathcal{L}_{\overline{Z}}^m$ and $\omega_{\overline{Z}}^m$ implies the equality of the sheaves $\mathcal{L}_{\overline{Z}}^j$ and $\omega_{\overline{Z}}^j$ for all j, and since all these sheaves are $\omega_{\overline{Z}}^m$ (and obviously $\mathcal{L}_{\overline{Z}}^m$)-reflexive, the isomorphisms

$$\mathcal{L}_{\overline{Z}}^j \simeq \mathcal{H}om(\mathcal{H}om(\mathcal{L}_{\overline{Z}}^j, \mathcal{L}_{\overline{Z}}^m), \mathcal{L}_{\overline{Z}}^m) \simeq \mathcal{H}om(\mathcal{H}om(\mathcal{L}_{\overline{Z}}^j, \omega_{\overline{Z}}^m), \omega_{\overline{Z}}^m)$$

yield, upon applying the functor v_*, the isomorphisms

$$\mathcal{L}_Z^j \simeq \mathcal{H}om(\mathcal{H}om(\mathcal{L}_Z^j, \mathcal{L}_Z^m), \mathcal{L}_Z^m) \simeq \mathcal{H}om(\mathcal{H}om(\mathcal{L}_Z^j, \omega_Z^m), \omega_Z^m)$$

proving **(iv)**. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

In the special case where the normalization is smooth, we have:

Corollary 2.10. *Let Z be a reduced complex space and $v : \overline{Z} \to Z$ its normalization with $\overline{Z}$ smooth. Then, Z is of type $\mathfrak{WR}^1$, and the sheaves $\mathcal{L}_Z^j$ are maximal Cohen-Macaulay for every integer $j \in \{0, 1, \cdots, m\}$. Moreover, the sheaves $\mathcal{L}_Z^j$ are ω_Z^m (resp. $\mathcal{L}_Z^m$, resp. $\mathcal{L}_Z^0$)-reflexive.*

Proof. By the preceding result, the sheaves $\mathcal{L}_Z^j$ ($0 \le j \le m$) all have depth at least two in Z, which is consequently of type $\mathfrak{WR}^1$. To see that they are maximal Cohen-Macaulay, we may assume Z is Stein and hence $\widehat{Z}$ is as well, and write, using Serre duality [Se],

$$\mathrm{H}_c^i(Z, \mathcal{L}_Z^j) \simeq \mathrm{H}_c^i(\widehat{Z}, \Omega_{\widehat{Z}}^j) \simeq \big(\mathrm{H}^{m-i}(\widehat{Z}, \Omega_{\widehat{Z}}^{m-j})\big)' = 0, \ \ \forall i \ne m, \ \forall j \in \{0, 1, \cdots, m\}.$$

Their $\mathcal{L}_Z^m$ and ω_Z^m-reflexivity is already established in the previous proposition. Their $\mathcal{L}_Z^0$-reflexivity follows from the $O_{\overline{Z}}$-reflexivity of the sheaves $\Omega_{\overline{Z}}^j$, which is easily deduced from the remark in the preceding result:

$$\mathcal{L}_Z^j = v_* \Omega_{\widehat{Z}}^j \simeq \mathcal{H}om(v_* \Omega_{\widehat{Z}}^{m-j}, v_* O_{\overline{Z}}) \simeq \mathcal{H}om(\mathcal{L}_Z^{m-j}, \mathcal{L}_Z^0)$$

and thus,

$$\mathcal{L}_Z^j \simeq \mathcal{H}om(\mathcal{H}om(\mathcal{L}_Z^j, \mathcal{L}_Z^0), \mathcal{L}_Z^0).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 2.11. **(i)** The sheaf $\mathcal{L}_Z^0$ always has depth at least two on any arbitrary reduced analytic space. Indeed, if Z is normal, the assertion is trivial since $O_Z \simeq \mathcal{L}_Z^0$. If not, it suffices to recall that this sheaf is the direct image under the normalization morphism of the structural sheaf (which has depth at least two).

(ii) Note that we always have

$$\mathcal{L}_Z^m \simeq \mathcal{H}om(\mathcal{L}_Z^0, \mathcal{L}_Z^m), \ \ \mathcal{L}_Z^0 \simeq \mathcal{H}om(\mathcal{L}_Z^m, \mathcal{L}_Z^m) \simeq \mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m)$$

and if Z is of type $\mathfrak{WR}^1$,

$$\mathcal{L}_Z^j \simeq v_* \omega_{\overline{Z}}^j \simeq \mathcal{H}om(\mathcal{L}_Z^{m-j}, \mathcal{L}_Z^m) \simeq \mathcal{H}om(\mathcal{L}_Z^{m-j}, \omega_Z^m)$$

Proposition 2.12. *Let Z be a reduced complex space. Then, the following are equivalent:*
(i) *Z is of type $\mathfrak{WR}^1$ and its singular locus has codimension at least two,*
(ii) *Z is of type $\mathfrak{WR}^2$.*

Proof. The implication **(i)** $\implies$ **(ii)** is evident since by [Proposition 2.5](), the sheaf $\mathcal{L}_Z^m$ has depth at least two and coincides, on the regular part of Z, with the sheaf ω_Z^m. These two sheaves thus coincide everywhere on Z.

Conversely, condition **(ii)** immediately implies that the sheaf $\mathcal{L}_Z^m$ has depth at least two. It remains to show that the singular locus $\Sigma := \mathrm{Sing}(Z)$ is of codimension at least two in Z. To see this, consider the short exact sequence

$$0 \longrightarrow O_Z \longrightarrow \mathcal{L}_Z^0 \longrightarrow \mathcal{K} \longrightarrow 0$$

where $\mathcal{K}$ is the coherent quotient sheaf whose support $\mathrm{Supp}(\mathcal{K})$ is exactly Σ. We may assume Σ reduced (and smooth if desired) of pure codimension 1 endowed with a natural embedding $i : \Sigma \hookrightarrow Z$.

Then, from the distinguished triangle

$$\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_Z^\bullet) \xrightarrow{\hspace{3cm}} \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^0, \mathcal{D}_Z^\bullet)$$
$$\overset{[1]}{\searrow} \qquad \swarrow$$
$$\mathcal{D}_Z^\bullet$$

we obtain the exact sequence $(\star)$

$$\cdots \longrightarrow \mathcal{H}^{-m}(\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_Z^\bullet)) \longrightarrow \mathcal{H}^{-m}(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^0, \mathcal{D}_Z^\bullet)) \longrightarrow \mathcal{H}^{-m}(\mathcal{D}_Z^\bullet)$$
$$\cdots \longleftarrow \mathcal{H}^{-m+1}(\mathcal{D}_Z^\bullet) \longleftarrow \mathcal{H}^{-m+1}(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^0, \mathcal{D}_Z^\bullet)) \longleftarrow \mathcal{H}^{-m+1}(\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_Z^\bullet))$$

We use then:

(a) if $\mathcal{F}$ is a coherent sheaf such that $\dim(\mathrm{Supp}(\mathcal{F})) < m$, then $\mathcal{H}^{-j}(\mathbb{R}\mathcal{H}om(\mathcal{F}, \mathcal{D}_Z^\bullet)) = 0$ for all $j > m$ since, for any Stein open subset U, $\mathrm{Ext}^{-m}(U; \mathcal{F}, \mathcal{D}_Z^\bullet) \simeq \big(\mathrm{H}_c^m(U, \mathcal{F})\big)'$ by [RR70] and $\mathrm{H}_c^m(U, \mathcal{F}) = 0$ (cf. [R]).

(b) the duality theorem for the morphism φ ([RRV71]) and the cohomology vanishing theorem ([GR70],[S]), gives the quasi-isomorphism $\mathcal{L}_Z^m = \mathbb{R}\varphi_* \Omega_{\hat{Z}}^m \simeq \mathbb{R}\mathcal{H}om(\mathbb{R}\varphi_* O_{\widetilde{Z}}, \mathcal{D}_Z^\bullet)[-m]$, and then a natural morphism $\mathcal{L}_Z^m \to \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^0, \mathcal{D}_Z^\bullet)[-m]$ which give us the injective morphism

$$\mathcal{L}_Z^m \to \mathcal{H}^{-m}(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^0, \mathcal{D}_Z^\bullet))$$

because it's a generic isomorphism of coherent sheaves with $\mathcal{L}_Z^m$ torsion free. But this implies, given our hypotheses, the isomorphisms

$$\mathcal{L}_Z^m \simeq \mathcal{H}^{-m}(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^0, \mathcal{D}_Z^\bullet)) \simeq \omega_Z^m$$

(c) the morphism $\mathcal{H}^{-m+1}(\mathbb{R}\mathcal{H}om(\mathcal{L}_Z^0, \mathcal{D}_Z^\bullet)) \to \mathcal{H}^{-m+1}(\mathcal{D}_Z^\bullet)$ is injective.

Indeed, let $\nu : \overline{Z} \to Z$ be the normalization of Z. Again by duality for the finite morphism ν, we have

$$\nu_*(\mathcal{D}_{\overline{Z}}^\bullet) \simeq \mathbb{R}\mathcal{H}om(\mathcal{L}_Z^0, \mathcal{D}_Z^\bullet)$$

and by considering the triangle induced by the trace between the dualizing complexes

$$\nu_*(\mathcal{D}_{\overline{Z}}^\bullet) \xrightarrow{\hspace{3cm}} \mathcal{D}_Z^\bullet$$
$$\overset{[1]}{\nwarrow} \qquad \swarrow$$
$$C^\bullet$$

in which we necessarily have $C^\bullet \simeq \mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_Z^\bullet)[1]$. The exact sequence $(\star)$ can then be written as

$$\mathcal{H}^{-m}(\nu_*(\mathcal{D}_{\overline{Z}}^\bullet)) \hookrightarrow \mathcal{H}^{-m}(\mathcal{D}_Z^\bullet) \longrightarrow \mathcal{H}^{-m}(C^\bullet) \longrightarrow \mathcal{H}^{-m+1}(\nu_*(\mathcal{D}_{\overline{Z}}^\bullet)) \longrightarrow \mathcal{H}^{-m+1}(\mathcal{D}_Z^\bullet) \longrightarrow \cdots$$

Since
$$\mathcal{H}^{-m}(v_*(\mathcal{D}_{\overline{Z}}^\bullet)) \simeq v_*\omega_{\overline{Z}}^m, \ \mathcal{H}^{-m}(\mathcal{D}_Z^\bullet) \simeq \omega_Z^m, \ \mathcal{H}^{-m}(C^\bullet) \simeq \mathrm{Coker}(v_*\omega_{\overline{Z}}^m \to \omega_Z^m)$$
and since $\overline{Z}$ is of type $\mathfrak{WR}^2$ because Z is, we have $v_*\omega_{\overline{Z}}^m \simeq \omega_Z^m$ (since $\mathcal{L}_{\overline{Z}}^m \simeq \omega_{\overline{Z}}^m$). Consequently, $\mathcal{H}^{-m}(C^\bullet) = 0$, and then the desired injectivity and the vanishing $\mathcal{H}^{-m+1}(\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_Z^\bullet)) = 0$.

We will show that this vanishing forces Σ to be of codimension at least two in Z which contradicts our hypothesis.

We have by duality $\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_Z^\bullet[-m+1]) \simeq i_*\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_\Sigma^\bullet[-m+1])$. As Σ is reduced, the regular part is a dense open subset of Σ. We can assume, in the first time, Σ smooth. Thus
$$\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_\Sigma^\bullet[-m+1]) \simeq \mathbb{R}\mathcal{H}om(\mathcal{K}, \Omega_\Sigma^{m-1})$$
and then
$$\mathcal{H}^{-m+1}(\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_Z^\bullet)) \simeq i_*\mathcal{H}om(\mathcal{K}, \Omega_\Sigma^{m-1}).$$

• If $\mathcal{K}$ is a torsion free sheaf on Σ, the vanishing of $\mathcal{H}om(\mathcal{K}, \Omega_\Sigma^{m-1})$ gives $\mathcal{K} = 0$ thanks to the injection of this sheaf in his bidual
$$\mathcal{K} \to \mathcal{H}om(\mathcal{H}om(\mathcal{K}, \Omega_\Sigma^{m-1}), \Omega_\Sigma^{m-1})$$
But this implies that Z is normal due to the isomorphism $O_Z \simeq \mathcal{L}_Z^0$ which contradicts our hypothesis on Σ assumed to be of pure codimension 1!

• in the general case, we note that the sheaf $\mathcal{H}om(\mathcal{K}, \Omega_\Sigma^{m-1})$ only accounts for the torsion-free part of $\mathcal{K}$ since if $\mathcal{T}$ is its torsion subsheaf, we have $\mathcal{H}om(\mathcal{K}, \Omega_\Sigma^{m-1}) \simeq \mathcal{H}om(\mathcal{K}/\mathcal{T}, \Omega_\Sigma^{m-1})$. In this case, we deduce that $\mathcal{K}$ must be a torsion sheaf, and then $\dim \mathrm{Supp}(\mathcal{K}) < \dim(\Sigma)$ which is impossible! Finally, we necessarily have $\dim(\Sigma) \leq m - 2$; thus, the singular locus has codimension at least two in Z.

For Σ not necessarily smooth, we can use:

★ the fact that $\mathcal{H}^0(\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_\Sigma^\bullet)[-m+1]) \simeq \mathcal{H}om(\mathcal{K}, \omega_\Sigma^{m-1})$ and argue as in the smooth case replacing Ω_Σ^{m-1} by ω_Σ^{m-1} or

★ by taking a local parametrization $f : \Sigma \to V$ (finite open surjective morphism Stein open set of $\mathbb{C}^{m-1}$) for which we have
$$f_*\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_\Sigma^\bullet[-m+1]) \simeq \mathbb{R}\mathcal{H}om(f_*\mathcal{K}, \Omega_V^{m-1})$$
and then
$$\mathcal{H}^{-m+1}(f_*\mathbb{R}\mathcal{H}om(\mathcal{K}, \mathcal{D}_\Sigma^\bullet)) \simeq \mathcal{H}om(f_*\mathcal{K}, \Omega_V^{m-1})$$
We reduce the problem to the smooth case because the finiteness and openness of f the sheaf $\mathcal{K}$ is torsion free if and only if $f_*(\mathcal{K})$ is torsion free which easily deduced from:
$$f_*\mathcal{H}_T^0(\mathcal{K}) \subset \mathcal{H}_{f(T)}^0(f_*\mathcal{K}), \ f_*\mathcal{H}_{f^{-1}(T')}^0(\mathcal{K}) \simeq \mathcal{H}_{T'}^0(f_*\mathcal{K})$$
and the existence of the canonical surjective morphism $f^*f_*\mathcal{F} \to \mathcal{F}$ for any coherent sheaf (using the fact that for any nowhere dense subset T (resp. T') of Σ (resp. V) $f(T)$ (resp. $f^{-1}(T')$) are nowhere dense in V (resp. Σ, by openness of f).

We then arrive at the desired conclusion $\dim \mathrm{Supp}(\mathcal{K}) = \dim(\Sigma) \leq m - 2$. $\qquad \square$

Remark 2.13. It is no hard to see that Z is of type $\mathfrak{WR}^2$ if and only if the sheaf $\mathcal{L}_Z^m$ is ω_Z^m-reflexive and the singular locus of Z has codimension at least two.

• indeed by Proposition 2.12, we know that the singular locus of Z necessarily has codimension at least two. Since, by definition, Z is of type $\mathfrak{WR}^2$ if and only if the canonical injective morphism $\mathcal{L}_Z^m \to \omega_Z^m$ is bijective, the sheaf $\mathcal{L}_Z^m$ has depth at least two. Then, the ω_Z^m-reflexivity of $\mathcal{L}_Z^m$, i.e. the bijectivity of the canonical injective morphism
$$\mathcal{L}_Z^m \to \mathcal{H}om(\mathcal{H}om(\mathcal{L}_Z^m, \omega_Z^m), \omega_Z^m)$$

is immediate since both sheaves have depth at least two and coincide outside the singular locus which has codimension at least two (moreover, ω_Z^m is naturally ω_Z^m-reflexive).

• Conversely, the ω_Z^m-reflexivity of the sheaf $\mathcal{L}_Z^m$ implies that its depth is at least two. Since the singular locus has codimension at least two and $\mathcal{L}_Z^m$ coincides with the sheaf ω_Z^m (which has depth at least two), they are globally identified on Z

One of the consequences of ([KeSc], Theorem (1.2), Theorem (1.3))

Theorem 2.14. *([KeSc])*

Let Z be a complex space. Then the following are equivalent:
(i) *Z is of type $\mathfrak{WR}^2$*
(ii) *Z is of type $k - \mathfrak{WR}^2$ for every integer $k \in \{0, 1, \cdots, m\}$.*

from which immediately follows the

Corollary 2.15. *Let Z be a complex space. Then the following are equivalent:*
(i) *Z is of type $\mathfrak{WR}^1$*
(ii) *Z is of type $k - \mathfrak{WR}^1$ for every integer $k \in \{0, 1, \cdots, m\}$.*

Proof. The implication **(ii)**$\Longrightarrow$**(i)** is obviously trivial. Conversely, if Z is of type $\mathfrak{WR}^1$, by preservation of depth under finite direct image, its normalization $\nu : \overline{Z} \to Z$ is of type $\mathfrak{WR}^2$, and Theorem 2.14 gives the equality of sheaves $\mathcal{L}_{\overline{Z}}^j = \omega_{\overline{Z}}^j$ for all $j \leq k$, and then $\mathcal{L}_Z^j = \nu_* \omega_{\overline{Z}}^j$. $\square$

3. BEHAVIOR UNDER PROPER MODIFICATION AND BASE CHANGE OF THE CLASSES $\mathfrak{WR}^2$ AND $\mathfrak{WR}^1$

Lemma 3.1. *Let $\psi : X \to Y$ be a proper and generically finite morphism of reduced complex spaces of pure dimension m. Then, $\mathbb{R}^j \psi_* \mathcal{L}_X^m = 0, \quad \forall j \neq 0$ and $\psi_* \mathcal{L}_X^m = \mathcal{L}_Y^m$.*

Proof. It suffices to consider the commutative diagram

$$
\begin{array}{ccc}
\widetilde{X} & & \\
\varphi \downarrow & \searrow^{\tilde{\varphi}} & \\
X & \xrightarrow{\psi} & Y
\end{array}
$$

in which φ is a resolution of singularities of X which is, therefore, also a desingularization of Y, and the Leray spectral sequence

$$\mathbb{R}^i \psi_* \mathbb{R}^j \varphi_* \Omega_{\widetilde{X}}^m \Longrightarrow \mathbb{R}^{i+j} \tilde{\varphi}_* \Omega_{\widetilde{X}}^m$$

degenerates because, by virtue of the Grauert-Riemenschneider theorem [GR70] or Takegoshi [Ta], we have $\mathbb{R}^j \varphi_* \Omega_{\widetilde{X}}^m = 0$ and $\mathbb{R}^j \tilde{\varphi}_* \Omega_{\widetilde{X}}^m = 0$ for all j non-zero. Thus, $\mathbb{R}^i \psi_* \mathcal{L}_X^m = 0$ for all $i \neq 0$ and $\psi_* \mathcal{L}_X^m = \psi_* (\varphi_* \Omega_{\widetilde{X}}^m) = \tilde{\varphi}_* \Omega_{\widetilde{X}}^m = \mathcal{L}_Y^m$. $\square$

The following results are three variants of the same result, namely the stability of the $\mathfrak{WR}^1$ property under proper modification.

Lemma 3.2. *Let Z be a reduced complex analytic space and T a subspace of Z. Let $\Theta : \widetilde{Z} \to Z$ be a proper modification of Z with center T (with preimage $\widetilde{T}$). Then, if Z is of type $\mathfrak{WR}^1$, $\widetilde{Z}$ is also of type $\mathfrak{WR}^1$.*

Proof. Note that the problem is local on Z, and the difficulty lies in studying the depth near points of the subset $\widetilde{T} \cap \mathrm{Sing}(\widetilde{Z}) \cap \Theta^{-1}(\mathrm{Sing}(Z))$. We may assume Θ is projective, since otherwise it can always be dominated by a blow-up. More precisely, for any Stein open subset U of Z, there exists a blow-up

with center $T \cap U$, $\varphi : V \to U$, and a blow-up of $\widetilde{U} := \Theta^{-1}(U)$ with center $\widetilde{T} \cap \widetilde{U}$, $\eta : V \to \widetilde{U}$, such that the following diagram commutes:

$$
\begin{array}{ccc}
V & \xrightarrow{\ \eta\ } & \widetilde{U} \\
& {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\Theta} \\
& & U
\end{array}
$$

If Θ is a projective morphism, there exists a coherent sheaf $\mathcal{F}$ on Z whose projectivization $\mathbb{P}(\mathcal{F})$ admits the factorization:

$$
\begin{array}{ccc}
\widetilde{Z} & \xrightarrow{\ \sigma\ } & \mathbb{P}(\mathcal{F}) \\
& {\scriptstyle\Theta}\searrow & \downarrow{\scriptstyle p} \\
& & Z
\end{array}
$$

where σ is an embedding and p is the canonical projection. We then reduce to the following situation:

$$
\begin{array}{ccc}
\widetilde{W} & \xrightarrow{\ \sigma\ } & U \times \mathbb{P}_N(\mathbb{C}) \\
& {\scriptstyle\Theta}\searrow & \downarrow{\scriptstyle p} \\
& & U
\end{array}
$$

Since p is a proper projection, we have:

$$
\mathrm{depth}(\mathcal{L}_Z^m) = \mathrm{depth}(p_*(\sigma_*\mathcal{L}_{\widetilde{Z}}^m)) \leq \mathrm{depth}(\sigma_*\mathcal{L}_{\widetilde{Z}}^m) = \mathrm{depth}(\mathcal{L}_{\widetilde{Z}}^m)
$$

More generally, if the direct image of a coherent sheaf under a proper and flat morphism has depth at least two, the sheaf itself necessarily has depth at least two.

Indeed, let $\pi : X \to Y$ be a proper, flat, and surjective morphism of reduced complex spaces, and $\mathcal{F}$ a coherent sheaf on X whose direct image $\pi_*\mathcal{F}$ has depth at least two. Since π is, in particular, open, it maps any open neighborhood of a given point $x \in X$ to an open neighborhood of $y := \pi(x)$. The coherent sheaf $\pi_*\mathcal{F}$, being of depth at least two by hypothesis, admits at least two holomorphic function germs f_y and g_y at y forming a regular sequence for the stalk $(\pi_*\mathcal{F})_y$. Taking representatives of f and g on a sufficiently small open neighborhood of y, the openness of π allows lifting f and g to a regular sequence for $\mathcal{F}$ near any point of the fiber $\pi^{-1}(y)$. Since x lies in this fiber, the conclusion follows (note that the openness of the morphism suffices to reach the conclusion; cf. [Fi], p.154). $\qquad\square$

Lemma 3.3. *Let $\varphi : X \to Y$ be a proper modification of reduced complex spaces, with Y of pure dimension m. Then, if Y is of type $\mathfrak{W}\mathfrak{R}^1$, X is also of type $\mathfrak{W}\mathfrak{R}^1$.*

Proof. Consider the commutative diagram:

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ \widetilde{\varphi}\ } & \widetilde{Y} \\
{\scriptstyle\widetilde{\psi}}\downarrow & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\psi} \\
X & \xrightarrow{\ \varphi\ } & Y
\end{array}
$$

where $\widetilde{Y}$ is a desingularization of Y, and $\widetilde{X}$ is a desingularization of the strict transform of X (i.e., the irreducible components of the fiber product $X \times_Y \widetilde{Y}$ that surject onto $\widetilde{Y}$). We also consider the two Leray spectral sequences:

$$
\mathrm{E}_2^{i,j} := \mathbb{R}^i\psi_*\mathbb{R}^j\widetilde{\varphi}_*O_{\widetilde{X}}, \quad {}'\mathrm{E}_2^{i,j} := \mathbb{R}^i\varphi_*\mathbb{R}^j\widetilde{\psi}_*O_{\widetilde{X}}
$$

converging to $\mathbb{R}^{i+j}\rho_*O_{\widetilde{X}}$.

Since $\mathrm{E}_2^{i,j} = 0$ for all $j \neq 0$, we have:

$$
\mathbb{R}^i\psi_*O_{\widetilde{Y}} \simeq \mathbb{R}^i\rho_*O_{\widetilde{X}}
$$

because $\widetilde{\varphi}_* O_{\widetilde{X}} \simeq O_{\widetilde{Y}}$ by normality.

Grauert's direct image theorem shows that $\Sigma_j := \mathrm{Supp}(\mathbb{R}^j \widetilde{\psi}_* O_{\widetilde{X}})$ are analytic subsets of dimension at most $m - j - 1$, and thus:
$$'E_2^{i,j} = 0 \quad \forall i + j \geq m.$$

We then deduce:
$$'E_2^{0,m-1} \simeq 'E_3^{0,m-1} \simeq \cdots = 'E_\infty^{0,m-1}.$$

By definition, in view of the surjective morphism[3]:
$$\mathbb{R}^{m-1} \rho_* O_{\widetilde{X}} \twoheadrightarrow 'E_\infty^{0,m-1},$$

we note that in general, we only have injections:
$$'E_\infty^{0,m-1} \hookrightarrow \cdots \hookrightarrow 'E_3^{0,m-1} \hookrightarrow 'E_2^{0,m-1},$$

and thus:
$$H^{m-1} \twoheadrightarrow 'E_\infty^{0,m-1} \hookrightarrow 'E_2^{0,m-1}$$

for the abutment H^{m-1}.

In our case, we obtain a surjective lateral morphism:
$$\mathbb{R}^{m-1} \rho_* O_{\widetilde{X}} \twoheadrightarrow 'E_2^{0,m-1} = \varphi_* \mathbb{R}^{m-1} \widetilde{\psi}_* O_{\widetilde{X}}.$$

We now use our hypotheses to see that:
$$\mathbb{R}^{m-1} \psi_* O_{\widetilde{Y}} \simeq \mathbb{R}^{m-1} \rho_* O_{\widetilde{X}} = 0,$$

and thus $\varphi_* \mathbb{R}^{m-1} \widetilde{\psi}_* O_{\widetilde{X}} = 0$.

Since the coherent sheaf $\mathbb{R}^{m-1} \widetilde{\psi}_* O_{\widetilde{X}}$ has support of dimension at most 0, we may assume, after sufficient localization, that it is reduced to a point x_0 in the singular locus $\mathrm{Sing}(X)$. Then, by definition, we have the equalities of finite-dimensional vector spaces:
$$(\mathbb{R}^{m-1} \widetilde{\psi}_* O_{\widetilde{X}})_{x_0} = \Gamma(X, \mathbb{R}^{m-1} \widetilde{\psi}_* O_{\widetilde{X}}) = \Gamma(Y, \varphi_* \mathbb{R}^{m-1} \widetilde{\psi}_* O_{\widetilde{X}}),$$

which, in view of the vanishing of $\varphi_* \mathbb{R}^{m-1} \widetilde{\psi}_* O_{\widetilde{X}}$, implies $\mathbb{R} \widetilde{\psi}_* O_{\widetilde{X}} = 0$. We conclude that X is of type $\mathfrak{WR}^1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can point out that the flatness assumption required in (Theorem 4.1, p.71) of [AS84] is not necessary to ensure the stability of the class $\mathfrak{WR}^1$ under finite morphisms.

Lemma 3.4. *Let $f : X \to Y$ be finite surjective morphism between reduced complex spaces. If X is of $\mathfrak{WR}^1$ type then so is Y.*

Proof. This fact is very simple to proving if we use the "depth"-characterization because, for a coherent sheaf $\mathcal{F}$ on X, we have $\mathrm{depth}(\mathcal{F}) \geq 2 \implies \mathrm{depth}(f_* \mathcal{F}) \geq 2$.

Through the use of desingularizations, this is not as straightforward. Consider the commutative diagramm

---

[3]These are general considerations for spectral sequences $E_r^{i,j}$ in the first quadrant, stabilized with a finite filtration $0 \subseteq F^n H^n \subseteq \cdots \subseteq F^1 H^n \subseteq H^n$ and converging to H, written as $E_r^{i,j} \implies H^{i+j}$. Then $E_\infty^{i,j} := F^i(H^{i+j})/F^{i+1}(H^{i+j}) = \mathrm{Gr}^i(H^{i+j})$.

Where φ is a desingularization of Y, X' the fiber product $X \times_Y Y'$, ν the normalization of the irreducibles components of X' and $\widetilde{X'}$ a desingularization of $\overline{X'}$.

The Leray spectrale sequence

$$E_2^{i,j} := \mathbb{R}^i \beta_* \mathbb{R}^j \alpha_* O_{\widetilde{X'}} \implies \mathbb{R}^{i+j}(\varphi' \circ \varphi")_* O_{\widetilde{X'}}$$

give us (by the same arguments used below)

$$'E_2^{m-1,0} \simeq 'E_3^{m-1,0} \simeq \cdots = 'E_\infty^{m-1,0}$$

and then an injective morphism

$$\mathbb{R}^{m-1} \beta_* \alpha_* O_{\widetilde{X'}} \hookrightarrow \mathbb{R}^{m-1}(\varphi' \circ \varphi")_* O_{\widetilde{X'}}.$$

But, by asumption and because $\widetilde{X'}$ is a resolution of singularities of X, $\mathbb{R}^{m-1}(\varphi' \circ \varphi")_* O_{\widetilde{X'}} = 0$. Thus, by normality of $\overline{X'}$,

$$\mathbb{R}^{m-1} \beta_* \alpha_* O_{\widetilde{X'}} = \mathbb{R}^{m-1} \beta_* O_{\overline{X'}} = \mathbb{R}^{m-1} \varphi'_* \nu_* O_{\overline{X'}} = 0$$

And obviously $f_* \mathbb{R}^{m-1} \varphi'_* \nu_* O_{\overline{X'}} = \mathbb{R}^{m-1} \varphi_* f'_* \nu_* O_{\overline{X'}} = 0$ As the finite morphism $h := f' \circ \nu$ is equidimensional over a smooth base (and then open surjective) we have a trace map $h_* O_{\overline{X'}} \to O_{Y'}$. It is easy to deduce the result since the sheaf $O_{Y'}$ is a direct factor of $h_* O_{\overline{X'}}$.   □

The following result can be deduced from Lemme 3.3, but we provide an independent proof, recalling (cf. [Fi],§3.6, [Si], I) that for any morphism $\pi : X \to Y$ of complex analytic spaces of dimensions m and r respectively, the degeneracy loci:

$$\mathrm{Deg}^k(\pi) := \{x \in X : \dim X_{\pi(x)} > k\}$$

are analytic. If X and Y are reduced and the morphism is generically open such that every irreducible component X_j of X maps to a single irreducible component Y_j of Y, the subsets:

$$F_j := \{x \in X_j : \dim X_{\pi(x)} > \dim X_j - \dim Y_j\}, \quad F := \bigcup_j F_j,$$

and $\pi(F)$ are thin in Y (cf. [[Si], Lemma 1.1]). If n is the generic dimension of π, we have a stratification:

$$\cdots \subset \mathrm{Deg}^{k+n+1}(\pi) \subset \mathrm{Deg}^{k+n}(\pi) \subset \cdots \subset \mathrm{Deg}^n(\pi) = F,$$

and a family of $(n+k)$-equidimensional morphisms:

$$\pi^k : \mathrm{Deg}^{k+n-1}(\pi) \setminus \mathrm{Deg}^{k+n}(\pi) \to Y.$$

Moreover, every point of $\mathrm{Deg}_k(\pi) := \mathrm{Deg}^{k+n-1}(\pi) \setminus \mathrm{Deg}^{k+n}(\pi)$ admits an open neighborhood U in $\mathrm{Deg}_k(\pi)$ such that $\pi^k(U)$ is locally analytic of dimension $\dim(\mathrm{Deg}_k(\pi)) - (n+k) < r$.

Lemma 3.5. *Let $\varphi : X \to Y$ be a proper modification of reduced complex spaces, with Y of pure dimension m. If for every subspace Σ of codimension at least two in Y,*

$$\mathcal{H}^i_\Sigma(\mathcal{L}^m_Y) = 0 \quad for\ i = 0, 1,$$

then the sheaf $\mathcal{L}^m_X$ satisfies the same property on X.

Proof. We wish to show that for every subspace $\widetilde{\Sigma}$ of codimension at least two in X,

$$\mathcal{H}^i_{\widetilde{\Sigma}}(\mathcal{L}^m_X) = 0 \quad \text{for}\ i = 0, 1.$$

For $i = 0$, this is clear since $\mathcal{L}^m_X$ is torsion-free. Moreover, if $\widetilde{\Sigma}$ lies outside the exceptional locus $E := \varphi^{-1}(T)$ (where T is the center of the modification, a subspace of codimension at least 1 in Y), the result follows immediately from the hypothesis on Y; this reduces the problem to considering only subspaces $\widetilde{\Sigma} \subsetneq E$.

Applying the functor φ_* to the injection:

$$\mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}^m_X) \hookrightarrow \mathcal{H}^1_{\varphi^{-1}(\varphi(\widetilde{\Sigma}))}(\mathcal{L}^m_X),$$

and knowing that:

$$\varphi_* \mathcal{H}^1_{\varphi^{-1}(\varphi(\widetilde{\Sigma}))}(\mathcal{L}_X^m) \simeq \mathcal{H}^1_{\varphi(\widetilde{\Sigma})}(\mathcal{L}_Y^m),$$

we obtain the injective morphism:

$$\varphi_* \mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m) \hookrightarrow \mathcal{H}^1_{\varphi(\widetilde{\Sigma})}(\mathcal{L}_Y^m).$$

Since φ is a dominant morphism (i.e., maps irreducible components of X to irreducible components of Y), we may assume X, Y, and T are irreducible. Then:

• If T has codimension at least two in Y, then $\varphi(\Sigma)$ will a fortiori have codimension at least two in Y, and thus $\varphi_* \mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m) = 0$.

• If T has pure codimension 1, the restriction of φ to E is generically finite, meaning there exists a closed subset T' of T with empty interior such that the restriction $\varphi : E \setminus \varphi^{-1}(T') \to T \setminus T'$ is open, finite, and surjective onto a normal base (i.e., a ramified covering of some degree k). In this case, $\varphi(\widetilde{\Sigma} \cap \varphi^{-1}(T'))$ and $\varphi(\widetilde{\Sigma} \cap (E \setminus \varphi^{-1}(T')))$ will have empty interior in T, the former because it is contained in T', the latter by purity of the dimension of $E \setminus \varphi^{-1}(T')$ (dimension formula for an open and surjective morphism; cf. [Fi],**p.143–145**).

In conclusion, we may assume $\varphi(\widetilde{\Sigma})$ has codimension at least two in Y. Thus, since by hypothesis $\mathcal{H}^1_{\varphi(\widetilde{\Sigma})}(\mathcal{L}_Y^m) = 0$, we have $\varphi_* \mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m) = 0$. It remains to verify that this implies the vanishing $\mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m) = 0$.

We may assume all spaces X, Y, T, and E are irreducible and reduced. Then:

• If the restriction φ' of φ to E is equidimensional over T, we have:

$$\varphi_* \mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m) = 0 \implies \mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m) = 0.$$

Indeed, let $\mathcal{F}$ denote the sheaf $\mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m)$, which is an O_X-module with support in E, precisely it is a quasi-coherent analytic sheaf . Assume φ' is equidimensional with fibers of pure dimension n[4]. By ([Fi], *corollary* p.138), φ' can be assumed open. In this case, the hypothesis $\varphi_* \mathcal{F} = 0$ implies that for every open subset U of T, $\Gamma(U, \varphi_* \mathcal{F}) = 0$, and thus in particular for opens of the form $\varphi(V)$. But $\Gamma(V, \mathcal{F}) \subset \Gamma(\varphi^{-1}(\varphi(V)), \mathcal{F}) = \Gamma(\varphi(V), \varphi_* \mathcal{F}) = 0$, and then varying V over a base of opens in E, we deduce that $\mathcal{F} = 0$.

Here is an another way to see it. First, we note that the sheaf $\mathcal{F} := \mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m)$ is coherent. Indeed, consider the short exact sequence

$$0 \to \mathcal{L}_X^m \to \omega_X^m \to \mathcal{K} \to 0$$

in which $\mathcal{K}$ is the quotient coherent sheaf supported on the singular locus of Z.

Then we have the long exact sequence of cohomology

$$0 \to \mathcal{H}^0_{\widetilde{\Sigma}}(\mathcal{K}) \to \mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{L}_X^m) \to \mathcal{H}^1_{\widetilde{\Sigma}}(\omega_X^m) \to \mathcal{H}^1_{\widetilde{\Sigma}}(\mathcal{K}) \to \cdots$$

and using the fact that:

★ $\mathcal{H}^0_{\widetilde{\Sigma}}(\mathcal{L}_X^m) = \mathcal{H}^0_{\widetilde{\Sigma}}(\omega_X^m) = 0$ because these sheaves are torsion free,

★ $\mathcal{H}^1_{\widetilde{\Sigma}}(\omega_X^m) = 0$ since $\widetilde{\Sigma}$ is two-codimensional in X and $\mathrm{depth}(\omega_X^m) \geq 2$,

★ and that $\mathcal{H}^0_{\widetilde{\Sigma}}(\mathcal{K}) = \mathcal{H}^0_{\widetilde{\Sigma} \cap \mathrm{Sing}(Z)}(\mathcal{K})$ is a coherent sheaf,

---

[4]It can locally be factored into a finite morphism followed by a projection around each $x \in \widetilde{\Sigma}$ (cf. [Fi], §3.7, for example):

$$W \xrightarrow[f]{\varphi'} V \times U \xrightarrow{q} V \ ,$$

where W is an open neighborhood of x, V a neighborhood of $\varphi'(x)$, and U a (relatively compact polydisc) open subset in some numerical space $\mathbb{C}^n$, reducing the assertion to the case of a finite, surjective, and open morphism.

we arrive at the desired conclusion.

As $\mathcal{F}$ is the zero sheaf if and only if, for every point x of E, the stalk $\mathcal{F}_x$ is zero, our problem is local on E in neighborhood of the support of $\mathcal{F}$. As φ' is equidimensionnel and open in each point x of E, we can consider a local factorization of φ'

$$
\begin{array}{ccc}
\text{E} & \xrightarrow{\ f\ } & T \times U \\
& {\scriptstyle \varphi'}\searrow & \downarrow{\scriptstyle q} \\
& & T
\end{array}
$$

where f is a finite, surjective and open morphism, q is the canonical projection, and U is a relatively compact Stein open subset of some numerical space $\mathbb{C}^N$.

then the result is a easy consequence of the equivalences

$$\varphi'_*\mathcal{F} = q_*(f_*\mathcal{F}) = 0 \iff f_*\mathcal{F} = 0 \iff \mathcal{F} = 0$$

due to the fact that for a Stein morphism of complex spaces $h : X \to Y$, the canonical morphism $h^*h_*\mathcal{G} \to \mathcal{G}$ is surjective for any coherent sheaf $\mathcal{G}$ on X.

• If φ' does not have constant fiber dimension, we stratify it using the degeneracy loci of φ'. More precisely, with the above notations, we consider:

$$\mathrm{Deg}^k(\varphi) := \{x \in \text{E} : \dim \text{E}_{\varphi(x)} > k\}.$$

Clearly, if $\Sigma \cap \text{E} = \emptyset$, we are back in the equidimensional case. Otherwise, letting $\widetilde{\Sigma}_k := \Sigma \cap \mathrm{Deg}_k(\varphi')$, we have:

$$\bigcup_k \mathcal{H}^1_{\widetilde{\Sigma}_k}(\mathcal{L}^m_X) = \mathcal{H}^1_{\widetilde{\Sigma} \cap \text{E}}(\mathcal{L}^m_X).$$

Since for every integer k:

$$\varphi'^k_*(\mathcal{H}^1_{\widetilde{\Sigma}_k}(\mathcal{L}^m_X)) = 0,$$

and φ'^k is equidimensional, the result follows. $\qquad\square$

Lemma 3.6. *Let $\varphi : X \to Y$ be a proper modification centered on the singular locus Σ of a reduced complex analytic space Y of pure dimension m. Let $\mathcal{F}$ be a coherent sheaf on X, a subsheaf of the coherent sheaf ω^m_X, such that the direct image $\varphi_*\mathcal{F}$ is a subsheaf of $\mathcal{L}^m_Y$. Then, $\mathcal{F}$ is a subsheaf of the sheaf $\mathcal{L}^m_X$.*

Proof. Since φ is a proper modification of a reduced space of pure dimension, X is also reduced of pure dimension. Since $\mathcal{F}$ is a subsheaf of ω^m_X, it is torsion-free and $\mathrm{Supp}(\mathcal{F}) = X$. The assertion will be proven if we show that every section of $\mathcal{F}$ extends to a holomorphic form on any desingularization of X. Since φ is a modification centered on the singular locus of Y, it suffices to consider the desingularization of Y, as it factors through φ in the commutative diagram

$$
\begin{array}{ccc}
& \widetilde{Y} & \\
{\scriptstyle \psi'}\swarrow & & \searrow{\scriptstyle \psi} \\
X & \xrightarrow[\ \ \varphi\ \ ]{} & Y
\end{array}
$$

It is then enough to use the definitions and the fact that $\varphi_*\mathcal{F} \subset \varphi_*\mathcal{L}^m_X$. $\qquad\square$

Proposition 3.7. *Let $\pi : Z \to S$ be a universally n-equidimensional morphism with S having singularities of type $\mathfrak{WR}^2$. Let $\nu : \widetilde{S} \to S$ be the normalization morphism and the induced base change diagram*

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\Theta} & Z \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \widetilde{S} & \xrightarrow{\nu} & S \end{array}$$

Then,

$$\widetilde{Z} \text{ is of type } \mathfrak{WR}^2 \text{ (resp.} \mathfrak{WR}^1) \Longleftrightarrow Z \text{ is of type } \mathfrak{WR}^2 \text{ (resp.} \mathfrak{WR}^1).$$

Proof. Since π is open, the total preimage $\widetilde{Z} := \widetilde{S} \times_S Z$ coincides with the strict transform.

• $\widetilde{Z}$ is of type $\mathfrak{WR}^1 \Longleftrightarrow Z$ is of type $\mathfrak{WR}^1$:

This equivalence is obvious since ν being a finite modification, the same holds for Θ. Consequently, $\mathcal{L}_Z^m = \Theta_* \mathcal{L}_{\widetilde{Z}}^m$ with depth preservation due to the finiteness of Θ.

• $\widetilde{Z}$ is of type $\mathfrak{WR}^2 \Longleftrightarrow Z$ is of type $\mathfrak{WR}^2$:

$\Longrightarrow$: Since ν is a finite modification centered on the singular locus of S, Θ is a modification centered on $\Sigma := \pi^{-1}(\mathrm{Sing}(S))$, which is of codimension at least two in Z because S is of type $\mathfrak{WR}^2$ and π is open with fibers of constant dimension.

But since $\widetilde{Z}$ is of type $\mathfrak{WR}^2$, the sheaf $\mathcal{L}_{\widetilde{Z}}^m$ has depth at least two, and consequently, $\mathcal{L}_Z^m = \Theta_* \mathcal{L}_{\widetilde{Z}}^m$ does as well. Moreover, by hypothesis, $\omega_{\widetilde{Z}}^m$ and $\mathcal{L}_{\widetilde{Z}}^m$ coincide globally on $\widetilde{Z}$ and thus on $\widetilde{Z} \setminus \Theta^{-1}(\Sigma)$. We deduce, therefore, that ω_Z^m and $\mathcal{L}_Z^m$ coincide on $Z \setminus \Sigma$. But since both have depth at least two, they must necessarily coincide globally on Z. Hence the conclusion.

$\Longleftarrow$: It is clear that $\widetilde{Z}$ is of type $\mathfrak{WR}^1$ due to the equality $\mathcal{L}_Z^m = \Theta_* \mathcal{L}_{\widetilde{Z}}^m$, the finiteness of Θ, and the assumption on Z, which requires $\mathcal{L}_Z^m$ to have depth at least two. On the other hand, since Θ is a modification, we have an injective direct image (defined in the sense of torsion-free currents) $\Theta_* \omega_{\widetilde{Z}}^m \to \omega_Z^m$ and an injective morphism $\mathcal{L}_Z^m = \Theta_* \mathcal{L}_{\widetilde{Z}}^m \to \Theta_* \omega_{\widetilde{Z}}^m$. Since Z is assumed to be of type $\mathfrak{WR}^2$, we deduce the isomorphism $\mathcal{L}_Z^m \simeq \Theta_* \omega_{\widetilde{Z}}^m \simeq \omega_Z^m$. But the short exact sequence

$$0 \longrightarrow \mathcal{L}_{\widetilde{Z}}^m \longrightarrow \omega_{\widetilde{Z}}^m \longrightarrow \mathcal{K} \longrightarrow 0$$

where $\mathcal{K}$ is the quotient sheaf, which is torsion and supported on the singular locus of $\widetilde{Z}$, gives us the short exact sequence

$$0 \longrightarrow \Theta_* \mathcal{L}_{\widetilde{Z}}^m \longrightarrow \Theta_* \omega_{\widetilde{Z}}^m \longrightarrow \Theta_* \mathcal{K} \longrightarrow 0$$

in which the first arrow is bijective according to what precedes. Hence, $\Theta_* \mathcal{K} = 0$, which, by virtue of the finiteness of Θ ensuring the surjectivity of the canonical morphism

$$\Theta^* \Theta_* \mathcal{K} \to \mathcal{K}$$

imposes $\mathcal{K} = 0$ and, consequently, the desired conclusion. $\qquad\square$

Proposition 3.8. *Let $\pi : Z \to S$ be an n-geometrically flat morphism with S having singularities of type $\mathfrak{WR}^2$. Let $\varphi : \widetilde{S} \to S$ be a proper modification and the commutative diagram induced by this base change*

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\tilde{\varphi}} & Z \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \widetilde{S} & \xrightarrow{\varphi} & S \end{array}$$

Then,

$$(i) \ \widetilde{Z} \text{ of type } \mathfrak{WR}^2 \implies Z \text{ of type } \mathfrak{WR}^2$$

$$(ii) \ \widetilde{Z} \text{ of type } \mathfrak{WR}^1 \iff Z \text{ of type } \mathfrak{WR}^1.$$

Proof. **(i)** $\widetilde{Z}$ of type $\mathfrak{WR}^2 \implies Z$ of type $\mathfrak{WR}^2$:

We deliberately restrict ourselves to proper modifications ν along the singular locus of S (the general case is strictly identical).

As already mentioned, the openness of π ensures the identification of the total and strict preimages. Since φ is a modification centered at the singular locus $\text{Sing}(S)$, $\tilde{\varphi}$ is a proper modification centered at $\Sigma := \pi^{-1}(\text{Sing}(S))$. Without any loss of generality, we may assume all spaces of pure dimension and S normal thanks to the Proposition 3.7, and that φ is a desingularization morphism.

Note that the morphism π is then geometrically flat, meaning that its fibers can be endowed with cycle structures by assigning suitable multiplicities (cf. [B1],[BM], [BM2], [K1], [K2]); this is a notion of local nature on the source and stable under arbitrary base change. This holds, for example, for any equidimensional morphism over a normal base. Regarding our purpose, the subtleties related to this notion are not used, as we only need to preserve the openness and the constancy of the fiber dimension. The morphism $\tilde{\pi}$, obtained by base change, is also geometrically flat (hence open) with fibers of pure dimension n.

The local nature of the problem allows us to reduce to a diagram of local parameterizations of the form (with some abuse of notation),

$$
\begin{array}{ccc}
\widetilde{Z} & \xrightarrow{\tilde{\varphi}} & Z \\
\tilde{f} \downarrow & & \downarrow f \\
\widetilde{S} \times U & \xrightarrow[\varphi' := \varphi \otimes Id]{} & S \times U
\end{array}
$$

Our assertion will be proven if we show that every section ξ of the sheaf ω_Z^m is a section of the sheaf $\mathcal{L}_Z^m$.

To do this, we use the trace morphism induced by f, which, for ξ, yields the section $\text{Tr}_f(\xi)$ of the sheaf $\omega_{S \times U}^m$. But since S is of type $\mathfrak{WR}^2$, this section is, in fact, a section of the sheaf $\mathcal{L}_{S \times U}^m$. Since the trace commutes with any base change, we have

$$\text{Tr}_{\tilde{f}}(\tilde{\varphi}^*(\xi)) = \varphi'^* \text{Tr}_f(\xi)$$

with $\varphi'^*(\text{Tr}_f(\xi))$ naturally defining a section of the sheaf $\Omega_{\widetilde{S} \times U}^m$.

Thus, the meromorphic form $\tilde{\varphi}^*(\xi)$, a section of the sheaf $\tilde{\varphi}^*(\omega_Z^m)$, admits holomorphic traces relative to this type of local parameterizations (pullback of local parameterizations on Z). To invoke the universal property of the sheaf $\omega_{\widetilde{Z}}^m$, we would need to ensure that $\tilde{\varphi}^*(\xi)$ indeed defines a section of $\omega_{\widetilde{Z}}^m$ and, therefore, verify that its actual poles (excluding virtual poles) lie within $\text{Sing}(\widetilde{Z})$. If this is the case, the analyticity of these traces for arbitrary meromorphic forms of ω_Z^m provides us with a pullback morphism

$$\tilde{\varphi}^*(\omega_Z^m) \to \omega_{\widetilde{Z}}^m$$

Since $\widetilde{Z}$ is of type $\mathfrak{WR}^2$, this arrow can also be written as $\tilde{\varphi}^*(\omega_Z^m) \to \mathcal{L}_{\widetilde{Z}}^m$ which, upon applying the direct image functor $\tilde{\varphi}_*$ and composing with the canonical and injective morphism $\omega_Z^m \to \tilde{\varphi}_* \tilde{\varphi}^*(\omega_Z^m)$, yields the morphism

$$\omega_Z^m \to \mathcal{L}_Z^m$$

necessarily injective since generically bijective between torsion-free sheaves. Knowing that $\mathcal{L}_Z^m$ is a subsheaf of ω_Z^m, we deduce the desired bijectivity.

Let us now show that the form $\tilde{\xi} := \tilde{\varphi}^*(\xi)$ is indeed a section of the sheaf $\omega_{\widetilde{Z}}^m$.

Since it is obviously polar in $\tilde{\varphi}^{-1}(\mathrm{Sing}(Z))$, everything reduces to studying the position of the subspaces $\tilde{\varphi}^{-1}(\mathrm{Sing}(Z))$ and $\mathrm{Sing}(\widetilde{Z})$, while convincing oneself that the true poles are contained in their intersection. We then distinguish the two fundamental cases:

- $\tilde{\varphi}^{-1}(\mathrm{Sing}(Z)) \cap \mathrm{Sing}(\widetilde{Z}) = \emptyset$:

This amounts to saying that $\tilde{\varphi}^{-1}(\mathrm{Sing}(Z)) \subset \mathrm{Reg}(\widetilde{Z})$ and, consequently, we find ourselves in the situation of a meromorphic form on a smooth variety, hence with poles in a certain divisor (since in codimension two, holomorphic extension is automatic), with this trace holomorphy condition translating into an $\mathbf{L}^2$ growth condition that guarantees global holomorphic extension[5] thanks to ([Gri76], **Lemma 2.1, p.349-352**). It follows that the poles located in $\mathrm{Reg}(\tilde{Z})$ are purely virtual since the form extends holomorphically there; this brings us back to the essential case.

- $\tilde{\varphi}^{-1}(\mathrm{Sing}(Z)) \cap \mathrm{Sing}(\widetilde{Z}) \neq \emptyset$:

In this case, it is, a fortiori, polar in $\mathrm{Sing}(\widetilde{Z})$, which is of codimension at least two. Since $\omega_{\widetilde{Z}}^m$ is of depth at least two (and thus satisfies the Hartogs extension principle), $\tilde{\varphi}^*(\xi)$ extends to a global section of this sheaf.

We can therefore consider that the meromorphic form $\tilde{\varphi}^*(\xi)$ has poles entirely contained in $\mathrm{Sing}(\widetilde{Z})$. Hence, thanks to the trace characterization,

$$\mathrm{Tr}_{\tilde{f}}(\tilde{\varphi}^*(\xi)) \text{ holomorphic} \implies \tilde{\varphi}^*(\xi) \text{ a section of } \omega_{\widetilde{Z}}^m$$

Since $\widetilde{Z}$ is of type $\mathfrak{WR}^2$, this means that $\tilde{\varphi}^*(\xi)$ is a section of $\mathcal{L}_{\widetilde{Z}}^m$ and, thus, ξ a section of $\mathcal{L}_Z^m$; which shows that Z is indeed of type $\mathfrak{WR}^2$.

Note that if φ is an arbitrary proper modification in which case $\tilde{S}$ may possibly be singular, we will consider local parameterizations on $\widetilde{S}$ to obtain local parameterizations on $\widetilde{Z}$. Since trace formation is compatible with the composition of local parameterization morphisms, we are reduced to the previous case where $\widetilde{S}$ is smooth.

(ii) $\widetilde{Z}$ of type $\mathfrak{WR}^1 \iff Z$ of type $\mathfrak{WR}^1$:

- $\Longleftarrow$: This implication is given by Lemme 3.3.

- $\Longrightarrow$: As noted earlier, we may assume all bases to be normal (cf. Proposition 3.7). Let us revisit the base change diagram and complete it with the normalization Z_1 (resp. Z_2) of Z (resp. $\widetilde{Z}$):

in which φ_1' is a finite modification, $\pi \circ \varphi_1$, $\tilde{\pi} \circ \varphi_1'$, ψ_1, and ψ_2 are universally n-equidimensional by stability under base change of the openness properties and constancy of fiber dimension, and because equidimensionality over a normal base implies the openness of the morphism (in fact, they are even geometrically flat since equidimensional over normal bases).

---

[5]If $f : V \to U$ is a finite, surjective morphism and ξ a meromorphic form on V whose direct image in the sense of currents $f_*\xi$ is a $\overline{\partial}$-closed current, then the current defined by ξ is necessarily $\overline{\partial}$-closed.

Note that if $\overline{Z} := \widetilde{S} \times_S Z_1 = \widetilde{Z} \times_Z Z_1$ is normal, there is no need to normalize $\widetilde{Z}$ and the result follows from **(i)** since

$$\widetilde{Z} \text{ of type } \mathfrak{WR}^1 \iff \overline{Z} \text{ of type } \mathfrak{WR}^2 \implies Z_1 \text{ of type } \mathfrak{WR}^2 \iff Z \text{ of type } \mathfrak{WR}^1$$

In the general case, we have the above commutative diagram, justified by the universal property of normalization, which ensures that the normalization morphism $Z_2 \to \widetilde{Z}$ factors through $\overline{Z}$ since φ_1' is a finite modification.

We want to show that $\mathcal{L}_{Z_1}^m = \omega_{Z_1}^m$. To do this, we revisit the local factorization diagrams:

As in **(i)**, we show that $\tilde{\varphi}'^*(\xi)$ defines a section of the sheaf $\omega_{\overline{Z}}^m$ for any section ξ of $\omega_{Z_1}^m$.

Thanks to the projection formula for a proper morphism, we obtain the isomorphisms

$$\tilde{f}_*(\psi^*(\omega_{Z_1}^m)) = \tilde{f}_*(\alpha^*(\tilde{\varphi}'^*(\omega_{Z_1}^m))) \simeq \bar{f}_*(\tilde{\varphi}'^*(\omega_{Z_1}^m) \otimes \alpha_* O_{Z_2}) \simeq \bar{f}_*(\tilde{\varphi}'^*(\omega_{Z_1}^m) \otimes \mathcal{L}_{\overline{Z}}^0)$$

and, consequently,

$$\hat{\varphi}_* \tilde{f}_*(\psi^*(\omega_{Z_1}^m)) \simeq \hat{\varphi}_* \bar{f}_*(\tilde{\varphi}'^*(\omega_Z^m) \otimes \mathcal{L}_{\overline{Z}}^0) \simeq f_*(\tilde{\varphi}_*'(\varphi'^*(\omega_Z^m) \otimes \mathcal{L}_{\overline{Z}}^0)) \simeq f_*(\omega_{Z_1}^m)$$

since $\tilde{\varphi}_*' \mathcal{L}_{\overline{Z}}^0 = O_{Z_1}$ by the normality of Z_1. By composing with the trace morphism induced by f, we thus deduce a nontrivial morphism

Since S (like $\widetilde{S}$) is of type $\mathfrak{WR}^2$, this gives the morphism

$$\hat{\varphi}_* \tilde{f}_*(\psi^*(\omega_{Z_1}^m)) \to \mathcal{L}_{S \times U}^m$$

and, since $\hat{\varphi}$ is a modification, this imposes the existence of a morphism

$$\tilde{f}_*(\psi^*(\omega_{Z_1}^m)) \to \mathcal{L}_{\widetilde{S} \times U}^m = \omega_{\widetilde{S} \times U}^m$$

By construction, this morphism factors through the trace induced by $\tilde{f}$ and, consequently, by the universal property of $\omega_{Z_2}^m$, we obtain a morphism $\psi^*(\omega_{Z_1}^m) \to \omega_{Z_2}^m$ since it is known that, in general, any finite morphism $h : X \to Y$ of reduced complex spaces of pure dimension m induces, via relative duality, an isomorphism of coherent sheaves ([K5])

$$f_* \mathcal{H}om_{O_X}(\mathcal{F}, \omega_X^m) \simeq \mathcal{H}om_{O_Y}(f_* \mathcal{F}, \omega_Y^m)$$

However, since Z_2 is of type $\mathfrak{WR}^2$, this morphism can also be written as

$$\psi^*(\omega_{Z_1}^m) \to \mathcal{L}_{Z_2}^m$$

from which, as before, we deduce an injective morphism $\omega_{Z_1}^m \to \mathcal{L}_{Z_1}^m$ and, consequently, the identification

$$\mathcal{L}_{Z_1}^m \simeq \omega_{Z_1}^m$$

which of course means that Z_1 is of type $\mathfrak{WR}^2$. But since a reduced complex space is of type $\mathfrak{WR}^1$ if and only if its normalization is of type $\mathfrak{WR}^2$ according to Corollaire 2.10, it follows that Z is of type $\mathfrak{WR}^1$. $\qquad\square$

It is now easy to generalize [AS84] to the case of a singular base.

Corollary 3.9. *Let* $\pi : Z \to S$ *be a flat morphism of reduced complex spaces where the base and the fibers are of type* $\mathfrak{WR}^2$. *Then,* Z *is of type* $\mathfrak{WR}^2$.

Proof. Consider the base change diagram induced by a resolution of singularities $\nu : \widetilde{S} \to S$, where the base change diagram

$$\begin{array}{ccc} \widetilde{Z} & \stackrel{\theta}{\longrightarrow} & Z \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{S} & \stackrel{\nu}{\longrightarrow} & S \end{array}$$

is such that the strict preimage $\widetilde{Z}$ coincides with the total preimage because π is open. Flatness being preserved under base change, and the fibers over $\tilde{S}$ having the same singularity type as those over S (being analytically isomorphic), we reduce to the case treated in [AS84] and conclude by applying Proposition 3.8. $\qquad\square$

4. Transfer of properties $\mathfrak{R}$, $\mathfrak{WR}^1$, and $\mathfrak{WR}^2$ from the source to the base: Theorem 0.1

The referee has kindly indicated to me that some of the below results are known in an algebraic setting if we assume the projectivity of Z and S by purity of the morphism π.

Let $\pi : Z \to S$ be an n-geometrically flat morphism of reduced complex spaces of pure dimensions m and r, respectively. We will show that if Z has rational (resp. $\mathfrak{WR}^2$, resp. $\mathfrak{WR}^1$) singularities, then S has rational singularities (resp. $\mathfrak{WR}^2$, resp.$\mathfrak{WR}^1$).

(i) Z **is of type** $\mathfrak{R} \implies S$ **is of type** $\mathfrak{R}$.

To do this, we use Kempf's rationality criterion. First, we observe that:

a) Z **normal** $\implies S$ **normal.**

Since the problem is local on S and Z, and π is n-geometrically flat, we may assume, given a local factorization around a point z of Z:

$$\begin{array}{ccc} Z & \stackrel{f}{\longrightarrow} & S \times U \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle q} \\ & & S \end{array}$$

where f is finite, open, and surjective, q is the natural projection, and U is a relatively compact Stein open subset of a numerical space $\mathbb{C}^n$. Thus, it suffices to verify the assertion for f.

Consider a meromorphic function g that is locally bounded on $S \times U$. Clearly, its pullback also defines a locally bounded meromorphic function, which must then be holomorphic due to the normality of Z. However, since f is 0-geometrically flat, it induces a trace morphism $f_*O_Z \to O_{S \times U}$. Thus, $f_*(f^*(g))$ extends holomorphically over S. But since f is a ramified covering of some degree k, we have $f_*(f^*(g)) = k.g$. Therefore, $\frac{1}{k}f_*(f^*(g))$ is a holomorphic extension of g; hence, S is normal.

(b) Z **is CM** $\implies S$ **is CM.**

Let $\mathcal{D}_Z^\bullet$ (resp. $\mathcal{D}_{S \times U}^\bullet$) be the dualizing complex of Z (resp. $S \times U$). By hypothesis, $\mathcal{H}^j(\mathcal{D}_Z^\bullet) = 0$ for all integers $j \neq -m$, and by relative analytic duality for a finite morphism, we have

$$\mathbb{R}f_*\mathcal{D}_Z^\bullet = \mathbb{R}\mathcal{H}om(\mathbb{R}f_*O_Z, \mathcal{D}_{S \times U}^\bullet)$$

that is, $f_* \mathcal{D}_Z^\bullet = \mathbb{R}\mathcal{H}om(f_* O_Z, \mathcal{D}_{S \times U}^\bullet)$. It follows that the homology of the complex $\mathcal{D}_{S \times U}^\bullet$ is a direct summand of the homology of $f_* \mathcal{D}_Z^\bullet$. Indeed, since $O_{S \times U}$ is a direct summand of $f_* O_Z$, $\mathcal{D}_{S \times U}^\bullet$ is also a direct summand of $\mathbb{R}\mathcal{H}om(f_* O_Z, \mathcal{D}_{S \times U}^\bullet)$. By the degeneration of the Leray spectral sequence for f, we have the isomorphism $f_* \mathcal{H}^j(\mathcal{D}_Z^\bullet) \simeq \mathcal{H}^j(f_* \mathcal{D}_Z^\bullet)$, and thus the vanishing

$$\mathcal{H}^j(\mathcal{D}_{S \times U}^\bullet) = 0, \, \forall \, j \neq -m.$$

Hence, $S \times U$ is CM, and consequently, S is CM (since U is smooth).

An alternative approach would be to show that the structure sheaf O_S is MCM, characterized by the vanishing of the cohomology

$$\mathrm{H}_c^i(S, O_S) = 0, \, \forall \, i \neq r.$$

To verify this, it suffices to use the hypothesis

$$\mathrm{H}_c^i(Z, O_Z) = 0, \, \forall \, i \neq m$$

and the fact that $O_{S \times U}$ is a direct summand of $f_* O_Z$. Indeed, in this case, $\mathrm{H}_c^i(Z, O_Z) \simeq \mathrm{H}_c^i(S \times U, f_* O_Z)$, and we have

$$\mathrm{H}_c^i(S \times U, O_{S \times U}) = 0, \, \forall \, i \neq m := n + r.$$

Then, a Künneth formula yields the decomposition

$$\mathrm{H}_c^i(S \times U, O_{S \times U}) = \bigoplus_{i'+j'=i} \mathrm{H}_c^{i'}(S, O_S) \widehat{\otimes}_{\mathbb{C}} \mathrm{H}_c^{j'}(U, O_U).$$

Since $\mathrm{H}_c^{j'}(U, O_U) = 0$ for all $j' \neq n$, we deduce that

$$\mathrm{H}_c^{i'}(S, O_S) = 0, \, \forall \, i' \neq r,$$

which implies that S is CM.

Since S is normal and CM, the rationality of its singularities relies on the equality $\mathcal{L}_S^r = \omega_S^r$ by Kempf's criterion (cf. Proposition 2.1).

By hypothesis, we have the equality $\mathcal{L}_Z^m = \omega_Z^m$, which, by [KeSc], *theorem (1.2)*, implies the equalities $\mathcal{L}_Z^k = \omega_Z^k$ for all $k \leq m$; this is a profound result whose proof requires powerful tools.

Again, by restricting to a suitable neighborhood adapted to a cycle $|Z_{s_0}|$, we may assume the morphism is finite. Let ξ be a section of the sheaf ω_S^r, and let $\xi' := \xi|_{\mathrm{Reg}(S)}$ be its restriction to the regular part. Then, $\pi^*(\xi')$ defines an r-holomorphic form on the dense subset $\pi^{-1}(\mathrm{Reg}(S)) = Z \setminus \pi^{-1}(\mathrm{Sing}(S))$, whose complement $\pi^{-1}(\mathrm{Sing}(S))$ is a subspace of codimension at least two in Z. Since ω_Z^r has depth at least two (satisfying Hartog's extension principle), $\pi^*(\xi')$ extends naturally to a global section $\pi^*(\xi)$ of ω_Z^r. However, by [KeSc], *theorem (1.2)*, we have $\mathcal{L}_Z^r \simeq \omega_Z^r$. Thus, we construct a pullback

$$\pi^* \omega_S^r \to \mathcal{L}_Z^r.$$

This necessarily implies the desired isomorphism. Indeed, we have the diagram

$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\ \theta\ } & Z \\
{\scriptstyle \tilde{\pi}} \downarrow & & \downarrow {\scriptstyle \pi} \\
\tilde{S} & \xrightarrow[\ v\ ]{} & S
\end{array}
$$

where v is a desingularization and $\tilde{Z}$ is the desingularization of the fiber product $\tilde{S} \times_S Z$ (in fact, the desingularization of the reduction $(\tilde{S} \times_S Z)_{red}$).

As the problem is local on Z and thanks to the local factorization, we can assume π finite and surjective. We obtain the morphisms

$$\theta^* \pi^* \omega_S^r \to \theta^* \mathcal{L}_Z^r \to \Omega_{\tilde{Z}}^r$$

and then

$$v^* \omega_Z^r \to \tilde{f}_* \Omega_{\tilde{Z}}^r.$$

But the direct image in the sense of currents or the trace map yields the morphism $\tilde{f}_* \Omega_{\tilde{Z}}^r \to \Omega_{\tilde{S}}^r$, and consequently, the morphism $v^* \omega_Z^r \to \Omega_{\tilde{S}}^r$, from which we derive $\omega_S^r \to \mathcal{L}_S^r$, necessarily injective since generically bijective between torsion-free sheaves. Since $\mathcal{L}_S^r$ is a subsheaf of ω_S^r, the conclusion follows, and hence S has rational singularities.

(ii) Z of type $\mathfrak{WR}^2 \Longrightarrow S$ of type $\mathfrak{WR}^2$.

   Since the problem is local on Z and S, we use the local factorization from **(i) a)** and the reasoning from **(i) b)**, reducing the problem essentially to the case of a finite and geometrically flat morphism $\pi : Z \to S \times U$. Consider a section ξ of the sheaf $\omega_{S \times U}^m$. Since it is unclear whether the singular locus of S has codimension at least two, we cannot use the previous extension argument. However, we know that $f^*(\xi)$ is holomorphic on $Z \setminus f^{-1}(\mathrm{Sing}(S \times U))$ because ξ is holomorphic on $S \times U \setminus \mathrm{Sing}(S \times U)$. It is thus necessary to examine the intersection $\Sigma := f^{-1}(\mathrm{Sing}(S \times U))$ with $\mathrm{Sing}(Z)$. We observe that if $\Sigma \cap \mathrm{Sing}(Z) = \emptyset$ or, equivalently, $\Sigma \subset \mathrm{Reg}(Z)$, we are in the situation of Proposition 2.5, which guarantees that the dense open subset $f(\mathrm{Reg}(Z))$ has rational singularities and is thus of type $\mathfrak{WR}^2$. This leads us to focus primarily on the case where $\Sigma \cap \mathrm{Sing}(Z) \neq \emptyset$ (particularly when $\Sigma \subseteq \mathrm{Sing}(Z)$). However, since Z is of type $\mathfrak{WR}^2$, $\mathrm{Sing}(Z)$ has codimension at least two, and hence so does $\Sigma \cap \mathrm{Sing}(Z)$. Since any meromorphic form holomorphic in codimension two extends naturally to a section of ω_Z^m, the same holds for $f^*(\xi)$. At this stage, we can revisit the arguments from **(i)b)** to deduce that the pullback $f^* \omega_{S \times U}^m \to \omega_Z^m$ can also be written as $f^* \omega_{S \times U}^m \to \mathcal{L}_Z^m$, since Z is of type $\mathfrak{WR}^2$. As before, this forces the isomorphism $\mathcal{L}_{S \times U}^m \simeq \omega_{S \times U}^m$. Denoting the canonical projections by $p_1 : S \times U \to S$ and $p_2 : S \times U \to U$, we have the natural decompositions

$$\mathcal{L}_{S \times U}^m = p_1^*(\mathcal{L}_S^r) \otimes_{O_{S \times U}} p_1^*(\Omega_U^n) \simeq \mathcal{L}_S^r \widehat{\otimes}_{\mathbb{C}} \Omega_U^n$$

$$\omega_{S \times U}^m = p_1^*(\omega_S^r) \otimes_{O_{S \times U}} p_1^*(\Omega_U^n) \simeq \omega_S^r \widehat{\otimes}_{\mathbb{C}} \Omega_U^n,$$

from which we immediately derive the isomorphism $\mathcal{L}_S^r \simeq \omega_S^r$.

(iii) Z of type $\mathfrak{WR}^1 \Longrightarrow S$ of type $\mathfrak{WR}^1$.

We can argue globally by considering the commutative diagram

$$
\begin{array}{ccc}
\overline{Z} & & \\
\eta \downarrow & \searrow & \\
\widehat{Z} & \xrightarrow{\ \theta\ } & Z \\
\hat{\pi} \downarrow & & \downarrow \pi \\
\hat{S} & \xrightarrow{\ v\ } & S
\end{array}
$$

where the square is a base change diagram induced by the normalization v, and $\overline{Z}$ is the normalization of Z.

Since $\hat{S}$ is normal and the composition $\tilde{\pi} \circ \eta$ is equidimensional, it is necessarily geometrically flat. Since Z is of type $\mathfrak{WR}^1$, $\overline{Z}$ is of type $\mathfrak{WR}^2$. By **(ii)**, this implies that $\hat{S}$ is also of type $\mathfrak{WR}^2$, which means S is of type $\mathfrak{WR}^1$.

We can also argue locally by reusing the factorization from **(i) a)**, reducing the problem essentially to the case of a finite and geometrically flat morphism $\pi : Z \to S \times U$. The invariance of depth under finite morphisms and the existence of a trace morphism $\pi_* \mathcal{L}_Z^m \to \mathcal{L}_{S \times U}^m$ show, via the pullback $\mathcal{L}_{S \times U}^m \to \pi_* \mathcal{L}_Z^m$, that the sheaf $\mathcal{L}_{S \times U}^m$ is a direct summand of the sheaf $\pi_* \mathcal{L}_Z^m$, which has depth at least two. With the notation of **(ii)**, we have the decomposition

$$\mathcal{L}_{S \times U}^m = p_1^*(\mathcal{L}_S^r) \otimes_{O_{S \times U}} p_1^*(\Omega_U^n) \simeq \mathcal{L}_S^r \widehat{\otimes}_{\mathbb{C}} \Omega_U^n,$$

showing that $\mathrm{depth}(\mathcal{L}^m_{S\times U}) \geq 2 \iff \mathrm{depth}(\mathcal{L}^r_S) \geq 2$ since Ω^n_U is locally free. Hence, the conclusion follows.

5. Transfer of Properties $\mathfrak{WR}^2$ from Fibers to the Total Space: Theorem 0.2

We propose in what follows two different proofs of this result. The first is essentially a residue-type method used in [AS84], but with the notable difference that the lack of flatness requires some additional effort using Grauert's notion of active elements. The second, on the other hand, consists of reducing to a flat morphism by flattening the initial morphism to then apply [AS84] or its generalized form given in Corollaire 3.9. Depending on whether the morphism is proper or not, we appeal to global or local flattening theorems ([H1], [HLT], or [Si]).

Let $\pi : Z \to S$ be a universally n-equidimensional morphism (i.e., open with fibers of pure dimension n) whose fiber singularities, as well as those of S, are of type $\mathfrak{WR}^2$. Our goal is to show that Z has the same type of singularities.

The problem is local in nature on Z (and S), as it amounts to showing that for every point z of Z lying on the fiber $Z_s := \pi^{-1}(\pi(z))$, the following implication holds:

$$\mathcal{L}^n_{Z_s,z} \simeq \omega^n_{Z_s,z} \implies \mathcal{L}^m_{Z,z} \simeq \omega^m_{Z,z}.$$

Since the properties of being open, having fibers of constant dimension, and being of type $\mathfrak{WR}^1$ or $\mathfrak{WR}^2$ are stable under base change (the new fibers being analytically isomorphic to the original ones), and since, moreover, the $\mathfrak{WR}^1$ condition is preserved under blowups or proper modifications in general (cf. Lemme 3.2 and Lemme 3.3), we may always assume S is normal or even smooth thanks to Proposition 3.7 or Proposition 3.8. Let us also recall, as already mentioned several times, that the openness of the morphism π (and its transforms under successive base changes) ensures that strict and total transforms coincide.

5.1. **a) Method 1: Active Elements and Fundamental Exact Sequences.** Recall that an element g of an analytic algebra is said to be *active* if its residual class in its reduction is a *non-zero divisor*. We will primarily use ([Fi], **Lemma 2, p. 144**), which states that the pullback of an active element by an open morphism of analytic spaces (not necessarily reduced) with a pure-dimensional base remains an active element.

   (i) Smooth Base of Dimension 1. Let $s \in S$, U an open neighborhood of s, and $f \in \Gamma(U, \mathcal{O}_S)$ such that its germ f_s generates the maximal ideal $\mathcal{M}_{S,s}$ and is thus a non-zero divisor, hence active. Since π is an open morphism, we know from ([Fi], **Lemma 1, Lemma 2, p. 144**) that for every $x \in X$ and $s := \pi(x)$, any active element f of $\mathcal{M}_{S,s}$ lifts to an active element of $\mathcal{M}_{Z,z}$. We will loosely denote this element by $\tilde{f} := \pi^*(f)$. Clearly, for every $x \in \pi^{-1}(U)$, $Z_s|_{\pi^{-1}(U)} = \{\tilde{f} = 0\}$. Then, with some deliberate notational abuse omitting localization, multiplication by $\tilde{f}$ yields the short exact sequence

$$0 \longrightarrow \tilde{f}\mathcal{O}_Z \xrightarrow{\; \bullet \tilde{f} \;} \mathcal{O}_Z \longrightarrow i_{s*}\mathcal{O}_{Z_s} \longrightarrow 0$$

to which we apply the functors $\mathcal{H}om(-, \omega^m_Z)$ and $\mathcal{H}om(-, \mathcal{L}^m_Z)$, respectively, to obtain the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \omega^m_Z & \longrightarrow & \omega^m_{Z,\tilde{f}} & \longrightarrow & \mathcal{E}xt^1(i_{s*}\mathcal{O}_{Z_s}, \omega^m_Z) & \longrightarrow & 0 \\
  &   & \uparrow &   & \uparrow &   & \uparrow &   &   \\
0 & \longrightarrow & \mathcal{L}^m_Z & \longrightarrow & \mathcal{L}^m_{Z,\tilde{f}} & \longrightarrow & \mathcal{E}xt^1(i_{s*}\mathcal{O}_{Z_s}, \mathcal{L}^m_Z) & \longrightarrow & 0
\end{array}
$$

where $\omega^m_{Z,\tilde{f}} := \mathcal{H}om(\tilde{f}.\mathcal{O}_Z, \omega^m_Z)$ and $\mathcal{L}^m_{Z,\tilde{f}} := \mathcal{H}om(\tilde{f}.\mathcal{O}_Z, \mathcal{L}^m_Z)$.

We will complete this diagram by showing the existence of two natural and injective morphisms

$$\mathcal{E}xt^1(i_{s*}O_{Z_s}, \omega_Z^m) \to i_{s*}(\omega_{Z_s}^{m-1})$$

and

$$i_{s*}(\mathcal{L}_{Z_s}^{m-1}) \to \mathcal{E}xt^1(i_{s*}O_{Z_s}, \mathcal{L}_Z^m)$$

which will allow us to establish our result.

If Z is Cohen-Macaulay then this injective morphism $\mathcal{E}xt^1(i_{s*}O_{Z_s}, \omega_Z^m) \to i_{s*}\omega_{Z_s}^{m-1}$ is bijective, and we have a commutative diagram

$$
\begin{array}{ccc}
i_{s*}\omega_{Z_s}^{m-1} & \xrightarrow{\sim} & \mathcal{E}xt^1(i_{s*}O_{Z_s}, \omega_Z^m) \\
\uparrow && \uparrow \\
i_{s*}\mathcal{L}_{Z_s}^{m-1} & \hookrightarrow & \mathcal{E}xt^1(i_{s*}O_{Z_s}, \mathcal{L}_Z^m)
\end{array}
$$

legitimately prompting the question of whether the residue morphism of the first line induces a residue morphism on the second line.

• For the first, observe that the canonical morphism $\omega_Z^m[m] \to \mathcal{D}_Z^\bullet$ induces a natural arrow $\mathbb{R}\mathcal{H}om(i_{s*}O_{Z_s}, \omega_Z^m[m]) \to \mathbb{R}\mathcal{H}om(i_{s*}O_{Z_s}, \mathcal{D}_Z^\bullet)$, which, at the level of cohomology of degree $-(m-1)$, yields the desired morphism

$$\mathcal{H}^{-m+1}(\mathbb{R}\mathcal{H}om(i_{s*}O_{Z_s}, \omega_Z^m[m])) \to i_{s*}\mathcal{H}^{-m+1}(\mathcal{D}_{Z_s}^\bullet)$$

since, by construction, $i_{s*}\mathcal{D}_{Z_s}^\bullet = \mathbb{R}\mathcal{H}om(i_{s*}O_{Z_s}, \mathcal{D}_Z^\bullet)$ and $i_{s*}\mathcal{H}^{-m+1}(\mathcal{D}_{Z_s}^\bullet) = i_{s*}\omega_{Z_s}^{m-1}$.

To verify its injectivity, it suffices to examine the low-degree terms of the spectral sequence

$$E_2^{i,j} = \mathcal{E}xt^i(i_{s*}O_{Z_s}, \mathcal{H}^j(\mathcal{D}_Z^\bullet[-m])) \implies \mathcal{E}xt^{i+j}(i_{s*}O_{Z_s}, \mathcal{D}_Z^\bullet[-m])$$

$$
\begin{array}{ccc}
E_2^{1,0} & \hookrightarrow & \mathcal{E}xt^1(i_{s*}O_{Z_s}, \mathcal{D}_Z^\bullet[-m]) \\
\parallel\wr && \parallel\wr \\
\mathcal{E}xt^1(i_{s*}O_{Z_s}, \omega_Z^m) & \hookrightarrow & i_{s*}\omega_{Z_s}^{m-1}
\end{array}
$$

Note, in passing, that if Z is Cohen-Macaulay, examining the low-degree terms of this spectral sequence shows that the injective morphism $\mathcal{E}xt^1(i_{s*}O_{Z_s}, \omega_Z^m) \to i_{s*}\omega_{Z_s}^{m-1}$ is bijective, and we have a commutative diagram

$$
\begin{array}{ccc}
i_{s*}\omega_{Z_s}^{m-1} & \xrightarrow{\sim} & \mathcal{E}xt^1(i_{s*}O_{Z_s}, \omega_Z^m) \\
\uparrow && \uparrow \\
i_{s*}\mathcal{L}_{Z_s}^{m-1} & \hookrightarrow & \mathcal{E}xt^1(i_{s*}O_{Z_s}, \mathcal{L}_Z^m)
\end{array}
$$

in which the second vertical arrow is not necessarily injective!

• For the second, consider the commutative diagram

$$
\begin{array}{ccc}
\widetilde{Z}_s' \xrightarrow{\quad\alpha_s\quad} \widetilde{Z}_s \xrightarrow{\tilde{i}_s} \widetilde{Z} \\
\end{array}
$$

induced by the desingularization of Z, where $\widetilde{Z}_s$ denotes the total preimage of Z_s. We may assume the strict preimage $\widetilde{Z}_s'$, which is a union of irreducible components of this total preimage, is smooth.

We construct the morphism $i_{s*}(\mathcal{L}_{Z_s}^{m-1}) \rightarrow \mathcal{E}xt^1(i_{s*}O_{Z_s}, \mathcal{L}_Z^m)$ using elementary operations and compositions summarized in the diagram

$$
\begin{array}{ccccc}
i_{s*}\mathcal{L}_{Z_s}^{m-1} & = & i_{s*}\mathbb{R}\varphi_*^{"}\Omega_{\widetilde{Z}'_s}^{m-1} & = & \mathbb{R}\varphi_*\mathbb{R}\mathcal{H}om(\mathfrak{a}_*O_{\widetilde{Z}'_s}, \Omega_{\widetilde{Z}}^m)[1] \\
\downarrow & & & & \downarrow \\
& & & & \mathbb{R}\mathcal{H}om(\mathbb{R}\varphi_*\sigma_{s*}O_{\widetilde{Z}_s}, \mathbb{R}\varphi_*\Omega_{\widetilde{Z}}^m)[1] \\
& & & & \downarrow \\
\mathbb{R}\mathcal{H}om(i_{s*}O_{Z_s}, \mathcal{L}_Z^m)[1] & \longleftarrow & & & \mathbb{R}\mathcal{H}om(i_{s*}\mathbb{R}\varphi_*^{"}O_{\widetilde{Z}'_s}, \mathcal{L}_Z^m)[1]
\end{array}
$$

Recall that since $\widetilde{Z}'_s$ is smooth of codimension 1 in $\widetilde{Z}$, we have

$$\mathbb{R}\mathcal{H}om(\mathfrak{a}_*O_{\widetilde{Z}_s}, \Omega_{\widetilde{Z}}^m)[1] \simeq \mathcal{E}xt^1(\mathfrak{a}_*O_{\widetilde{Z}'_s}, \Omega_{\widetilde{Z}}^m) \simeq \mathfrak{a}_*\Omega_{\widetilde{Z}'_s}^{m-1}.$$

In particular, taking the degree 0 cohomology of the first vertical arrow yields the desired morphism

$$i_{s*}(\mathcal{L}_{Z_s}^{m-1}) \rightarrow \mathcal{E}xt^1(i_{s*}O_{Z_s}, \mathcal{L}_Z^m).$$

We then complete the previous two-line commutative diagram to obtain

$$
\begin{array}{c}
i_{s*}\mathcal{L}_{Z_s}^{m-1} \\
\downarrow{\scriptstyle\alpha} \\
0 \longrightarrow \mathcal{L}_Z^m \xrightarrow{\bullet f} \mathcal{L}_{Z,\tilde{f}}^m \longrightarrow \mathcal{E}xt^1(i_{s*}O_{Z_s}, \mathcal{L}_Z^m) \longrightarrow 0 \\
\quad\quad \downarrow \quad\quad {\scriptstyle\alpha'}\downarrow \quad\quad {\scriptstyle\beta}\downarrow \quad\quad {\scriptstyle\sigma} \\
0 \longrightarrow \omega_Z^m \xrightarrow{\bullet f} \omega_{Z,\tilde{f}}^m \longrightarrow \mathcal{E}xt^1(i_{s*}O_{Z_s}, \omega_Z^m) \longrightarrow 0 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad {\scriptstyle\gamma}\downarrow \\
i_{s*}\omega_{Z_s}^{m-1}
\end{array}
$$

and deduce that the bijectivity of the natural injection σ forces the surjectivity of γ and hence its bijectivity. This in turn implies the surjectivity of β and the injectivity of α. But this latter surjectivity entails that of α' and thus its bijectivity, which in turn ensures the bijectivity of the first vertical arrow namely the bijectivity of the natural injection of $\mathcal{L}_Z^m$ into ω_Z^m proving that Z is of type $\mathfrak{WR}^2$

(ii) Smooth base of dimension $r > 1$ and general case.

We proceed by induction on the dimension r of S, assuming the result holds for any base of dimension strictly less than r. The idea is to consider a smooth subspace of dimension $r-1$, perform a base change, apply the induction hypothesis to the resulting total space, and then view this subspace itself as a fiber at the origin (for example) of a family of hypersurfaces. These are classical arguments often used in such situations (cf. [AS84] or [Schn]).

Consider a point s_0 in S around which a local coordinate system $(s_1, \cdots, s_r)$ is given on a sufficiently small Stein open set V. Let $S' := \{s \in V : s_1(s) = \cdots = s_{r-1}(s) = 0\}$. Since π is open and s_r is a parameter near s_0 (thus inducing an active element on the local ring O_{S,s_0}), $\tilde{s}_r := \pi^*(s_r)$ is again an active element on the local ring of Z near any point with image s_0, by ([Fi], **Lemma 2, p. 144**). In the

Cartesian diagram of base change

$$
\begin{array}{ccc}
Z' := Z \times_S S' & \xrightarrow{\sigma'} & Z \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
S' & \xrightarrow{\sigma} & S
\end{array}
$$

Since π' is still flat with reduced fibers (analytically isomorphic to the fibers of π) and of type $\mathfrak{WR}^1$, the induction hypothesis applied to π' guarantees that Z' is also of type $\mathfrak{WR}^1$.

We then consider the canonical projection $p : \mathbb{C}^r \to \mathbb{C}^{r-1}$ sending $(z_1, \cdots, z_r)$ to $(z_1, \cdots, z_{r-1})$ and a local coordinate change isomorphism $h : V \to \mathbb{C}^r$. Then, if $S" := p(h(V))$, the composite morphism $\psi : Z \to S"$ is still geometrically flat, being open with fibers of dimension $n + 1$ over the smooth $S"$. Moreover, after possibly shrinking the data, we obtain the commutative diagram

which shows that the fiber over $\{0\}$ of this ψ is given by the analytic space $Z \times_{S"} \{0\}$. Now, the latter is analytically isomorphic to $Z' := Z \times_S S'$, which has singularities of type $\mathfrak{WR}^2$. We then conclude by applying the induction hypothesis. From this case, we deduce the general case of singular base of type $\mathfrak{WR}^2$ by applying Proposition 3.8

5.2. b) Method 2: By flattening the morphism.

Recall that a *local blow-up* of an analytic space Z consists of an open subset U of Z, a closed analytic subset Y of U, and the morphism $\sigma : V \to Z$ composed of the blow-up of Y in U and the inclusion morphism of U into Z. A sequence of local blow-ups of Z is a finite system $(U_\alpha, Z_\alpha, \sigma_\alpha)_{1 \le \alpha \le r}$ where each σ_α is a local blow-up and $Z_0 = Z$. Note that a local blow-up is generally never proper.

The known flattening theorems are primarily of two types: algebraic and geometric. In the following statements, the first two are of the first category, and the latter is of the second.

Theorem 5.1. *Global Flattening*[H1]: *Let $\pi : X \to Y$ be a morphism of complex analytic spaces with Y reduced. Let $\mathcal{F}$ be a coherent sheaf on X. Then, there exists a proper modification $\sigma : \widetilde{Y} \to Y$ composed of a finite sequence of blow-ups such that the strict transform $(\widetilde{\mathcal{F}}, \widetilde{\pi})$ of $(\mathcal{F}, \pi)$ is flat everywhere.*

Theorem 5.2. *Local Flattening*[HLT]: *Let $\pi : X \to Y$ be a morphism of complex analytic spaces with Y reduced. Let $y \in Y$ and L be a compact subset of the fiber $\pi^{-1}(y)$. Then, there exists a finite number of morphisms $\sigma_\alpha : Y_\alpha \to Y$, each composed of a finite number of local blow-ups whose centers lie in subsets with empty interior, such that:*

(i) *For each α, there exists a compact subset K_α of Y_α such that $\bigcup_\alpha \sigma_\alpha(K_\alpha)$ is a compact neighborhood of y in Y.*

(ii) *For each α, the strict transform $\pi_\alpha : X_\alpha \to Y_\alpha$ of π by σ_α is flat at points corresponding to L.*

Theorem 5.3. *Local Geometric Flattening*[Si]: *Let $\pi : X \to Y$ be a generically open morphism of reduced complex analytic spaces, and let K be a compact subset of X. Then, there exists a normal complex space Y', a holomorphic map $\sigma : Y' \to Y$, and a relatively compact open neighborhood V of K such that:*

(i) *If X' is the subspace of $Y' \times_Y V$ consisting of irreducible components over which the transform of π (by σ) remains generically open onto Y', then the strict transform $\pi' : X' \to Y'$ is open and surjective.*

(ii) *The induced map $\sigma' : X' \to V$ is semi-proper and surjective.*

(iii) *σ is generically open and generically finite.*

We have the diagram

$$
\begin{array}{ccc}
X' \rightarrowtail & Y' \times_Y V & \xrightarrow{\;\;\sigma'\;\;} X \\
 & \downarrow & \downarrow{\scriptstyle \pi} \\
 \pi' \searrow & Y' & \xrightarrow{\;\;\sigma\;\;} Y
\end{array}
$$

Note that σ is neither proper nor even semi-proper but preserves density by inverse image. Moreover, if E is the degeneracy locus of π (the union of subsets consisting of points where the fibers have dimension strictly greater than $\dim X - \dim Y$) and $N(Y)$ the non-normal locus of Y, then Y' is identified with the fiber product $Y' \times_Y V$ outside $\Sigma := \sigma^{-1}(\overline{\pi(E \cap V)} \cup N(Y))$,

$$
X' \setminus \pi'^{-1}(\Sigma) \simeq (Y' \setminus \Sigma) \times_Y V
$$

with $\sigma : Y' \setminus \Sigma \to Y$ (resp. $\pi' : X' \setminus \pi'^{-1}(\Sigma) \to X$) open.

This theorem follows from Theorem 5.2, as noted by Sibert in remark (**(6), p.274**).

Let us return to the proof of our theorem, for which we must distinguish the proper case from the non-proper case.

(i) The Proper Case.

This is an almost immediate corollary of Corollaire 3.9, as it suffices to flatten the morphism by applying Hironaka's global flattening theorem Theorem 5.1 to obtain the diagram:

$$
\begin{array}{ccc}
Z_1 & \xrightarrow{\;\Theta\;} & Z \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\
S_1 & \xrightarrow{\;\eta_1\;} & S
\end{array}
$$

noting that the strict preimage Z_1 coincides with the total preimage since π is open. Then, since S is of type $\mathfrak{WR}^2$ (and hence $\mathfrak{WR}^1$), S_1 is also of type $\mathfrak{WR}^1$ by virtue of Lemme 3.2 or Lemme 3.3.

We may then either normalize or desingularize the base S_1 to obtain the diagram:

$$
\begin{array}{ccccc}
Z_2 & \xrightarrow{\;\Theta_2\;} & Z_1 & \xrightarrow{\;\Theta_1\;} & Z \\
{\scriptstyle \pi_2}\downarrow & & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\
S_2 & \xrightarrow{\;\eta_2\;} & S_1 & \xrightarrow{\;\eta_1\;} & S
\end{array}
$$

where the morphism π_2 is still flat, with fibers and base of type $\mathfrak{WR}^2$. If we normalize, we apply Corollaire 3.9, and if we desingularize [AS84], we deduce that Z_2 is of type $\mathfrak{WR}^2$.

(ii) The Non-Proper Case. Here again, we may assume S is normal without loss of generality by applying Proposition 3.8 to the base change given by the normalization of S.

We then use either the local flattening theorem Theorem 5.2 or the geometric flattening theorem Theorem 5.3 from [Si]. We begin by noting the following:

• **Stability of the $\mathfrak{WR}^1$ Condition Under Local Blowup.** Let $\varphi : S' \to S$ be a local blowup, i.e., the composition of an open embedding $j : V \hookrightarrow S$ and a blowup $\sigma : S' \to V$ whose center lies in a nowhere dense subset of S. Clearly, a local blowup is never proper in general.

The stability under open restriction (which is a flat morphism) and under blowup (which is a proper modification to which we apply Lemme 3.2 or Lemme 3.3) will give us stability under an arbitrary local blowup.

Note that the inverse image $\mathcal{L}_S^m \to \varphi_* \mathcal{L}_{S'}^m$ naturally factors as $\mathcal{L}_S^m \to j_* \mathcal{L}_V^m \to \varphi_* \mathcal{L}_{S'}^m$, and that

$$\varphi_* \mathcal{L}_{S'}^m = j_* \mathcal{L}_V^m = j_* j^* \mathcal{L}_S^m \simeq \mathcal{L}_S^m$$

since the sheaf $\mathcal{L}_S^m$ has depth at least two, and the center of local blowups may be chosen within the singular locus of codimension at least two in S.

• **Preimage Under a Local Blowup.** With the same notation, we have the commutative diagram:

$$
\begin{array}{ccccc}
Z'' & \xrightarrow{\;\sigma'\;} & Z' & \xrightarrow{\;j'\;} & Z \\
\downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
S' & \xrightarrow{\;\sigma\;} & V & \xrightarrow{\;j\;} & S
\end{array}
$$

where $Z' := Z \times_S V$ and $Z'' := Z' \times_V S' \simeq Z \times_S S'$. Since π is open with fibers of constant pure dimension n, the same holds for the morphisms π' and π'', as these properties are preserved under any base change. Moreover, as already mentioned, the total and strict preimages coincide under these base changes, which is crucial for our purpose since the fibers will always remain analytically isomorphic (in general, on the strict preimage, the new fibers are isomorphic to the old ones only generically over the base).

• This type of diagram will simply be denoted, for any α, as:

$$
\begin{array}{ccc}
Z_\alpha & \xrightarrow{\;\sigma'_\alpha\;} & Z \\
\downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle \pi} \\
S_\alpha & \xrightarrow{\;\sigma_\alpha\;} & S
\end{array}
$$

following the notation of Theorem 5.2. The idea is to proceed by induction on the number of local blowups constituting σ_α to reduce to the case of a single blowup. To do this, we revisit the arguments developed in [Par] point by point, allowing us to essentially reduce to the case of a standard blowup by localizing sufficiently, taking for the compact $L := \{z\}$ a point in the fiber $\pi^{-1}(\pi(z))$.

We consider an open neighborhood U of $s := \pi(z)$ and a compact subneighborhood K of s in U. Since the sheaves $\mathcal{L}_Z^m$ and ω_Z^m are compatible with open inclusions, we first replace the data (π, Z, S) with $(\pi|_{\pi^{-1}(U)}, \pi^{-1}(U), U)$, noting that the conditions of Theorem 5.2 are satisfied for $L := \{z\}$, $\sigma_\alpha := \sigma_\alpha|_{\sigma_\alpha^{-1}(U)}$, $K_\alpha := K_\alpha \cap \sigma_\alpha^{-1}(K)$. The centers of the local blowups $\sigma_\alpha|_{\sigma_\alpha^{-1}(U)}$ are open subsets of the centers of the σ_α. Moreover, for a blowup $\sigma : S' \to S$, the strict preimages of the σ_α are compositions of local blowups with rare centers contained in or equal to certain centers of the σ_α.

If $v : S' \to S$ is a given blowup, we obtain, via base changes, the following diagrams:

$$
\begin{array}{ccccccc}
Z'_\alpha & & \xrightarrow{\qquad v'_\alpha \qquad} & & Z_\alpha & & \\
& \searrow & & & & \searrow & \\
\pi'_\alpha \downarrow & & Z' & \xrightarrow{\;v'\;} & \pi_\alpha & & Z \\
& & \downarrow & & & & \downarrow \\
S'_\alpha & \xrightarrow{\quad \pi'\quad} & & v_\alpha & S_\alpha & & \pi \\
& \searrow & \downarrow & & & \searrow & \\
\sigma'_\alpha & & S' & \xrightarrow{\;v\;} & \sigma_\alpha & & S
\end{array}
$$

where ν, ν', ν_α, and ν'_α are proper morphisms, the pair $(\sigma'_\alpha, \nu_\alpha^{-1}(K_\alpha))$ satisfies the conditions of the local flattening theorem for the morphism $\pi' : Z' \to S'$, and $L' := \nu'^{-1}(L)$.

Of course, this remains entirely valid if ν is a local blowup. We then begin the induction on the number of local blowups, essentially reducing—after sufficient localization—to the case of a single blowup, where we can apply Proposition 3.7.

We may bypass the inductive reasoning and its constraints by adopting the formulation of Theorem 5.3. For a given z_0 in Z, we consider a relatively compact open neighborhood U with closure $K := \overline{U}$. Then, denoting by V the relatively compact neighborhood containing K provided by the theorem, we have the commutative diagram:

$$
\begin{array}{ccc}
S' \times_S Z' & \xrightarrow{\ \sigma'\ } & Z \\
{\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
S' & \xrightarrow{\ \sigma\ } & S
\end{array}
$$

where $Z' := V \times_S S'$, S' is normal, and σ and σ' are open.

We address the question directly by revisiting the diagram of local parametrizations (again with simplified notation):

$$
\begin{array}{ccc}
Z' & \xrightarrow{\ \sigma'\ } & Z \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
S' \times U & \xrightarrow{\ \sigma\ } & S \times U
\end{array}
$$

where, by slight abuse of notation, we denote the induced morphisms by the same letters σ and σ'. We then easily adapt Proposition 3.8 to deduce that if Z' is of type $\mathfrak{WR}^2$, so is Z.

More precisely, if ξ is a section of the sheaf ω_Z^m, we have:

$$
\mathrm{Tr}_{f'}(\sigma'^*(\xi)) = \sigma^* \mathrm{Tr}_f(\xi)
$$

Then, since by hypothesis $\mathrm{Tr}_f(\xi)$ defines a section of the sheaf $\mathcal{L}_{S \times U}^m$, it follows that $\mathrm{Tr}_{f'}(\sigma'^*(\xi))$ defines a section of the sheaf $\mathcal{L}_{S' \times U}^m$. Since local blowups, like open morphisms, preserve density under inverse images, the meromorphic form $\sigma'^*(\xi))$ has its poles in a nowhere dense closed subset contained in the nowhere dense closed subset $\sigma'^{-1}(\mathrm{Sing}(Z))$.

As in Proposition 3.8, we show that only the poles contained in $\sigma'^{-1}(\mathrm{Sing}(Z)) \cap \mathrm{Sing}(Z')$ matter. Moreover, if $S' \to V$ is a local parametrization of S', it yields, by composition, the parametrization:

$$
Z' \underset{f'}{\overset{h}{\rightrightarrows}} S' \times U \xrightarrow{\ g\ } V \times U
$$

for which we have:

$$
\mathrm{Tr}_h(\sigma'^*(\xi)) = \mathrm{Tr}_g(\mathrm{Tr}_{f'}(\sigma'^*\xi))
$$

which is holomorphic since $\mathrm{Tr}_{f'}(\sigma'^*\xi)$ is a section of the sheaf $\mathcal{L}_{S' \times U}^m$, a subsheaf of $\omega_{S' \times U}^m$.

Thus, the form $\sigma'^*(\xi)$ defines a section of the sheaf $\omega_{Z'}^m$. But since Z' is of type $\mathfrak{WR}^2$, this shows that $\sigma'^*(\xi)$ is a section of the sheaf $\mathcal{L}_{Z'}^m$. We immediately deduce that ξ must necessarily be a section of the sheaf $\mathcal{L}_Z^m$ and, consequently,

$$
\mathcal{L}_Z^m \simeq \omega_Z^m
$$

meaning Z is of type $\mathfrak{WR}^2$

6. Transfer of properties $\mathfrak{WR}^1$ from Fibers to the Total Space: Theorem 0.3

Let $\pi : Z \to S$ be a universally n-equidimensional morphism whose base and fibers have singularities of type $\mathfrak{WR}^1$, and we aim to show that Z also possesses this property. To do this, we proceed in several steps, starting by noting that since the constancy of fiber dimension and openness are properties invariant under base change, we may assume S is smooth or normal by virtue of Proposition 3.7 and Proposition 3.8.

6.1. **The case where π is a reduced morphism.** Since the property of being reduced for the fibers is also invariant under base change and we may assume the base is smooth, we reduce to the study of a flat morphism with reduced fibers over a smooth base (since any open morphism over a smooth base with reduced fibers is necessarily flat ([GR-Ke], [Kan], [OKY], ([Fi], **p.157**)).

In this case, the source is also reduced[6].

In the following particular cases, Theorem 0.2 is an immediate corollary of Theorem 0.2:

⋆ If π is equinormalizable or admits a simultaneous normalization ([ChLi], [G], [C.T-Lê]), then it is induced by the normalization of Z, $\nu : \overline{Z} \to Z$, with the commutative diagram

in which the fiber $\bar{\pi}^{-1}(s)$ is the normalization of the fiber $\pi^{-1}(s)$.

Then, we have

$$\pi^{-1}(s) \text{ of type } \mathfrak{WR}^1 \iff \bar{\pi}^{-1}(s) \text{ of type } \mathfrak{WR}^2 \implies \overline{Z} \text{ of type } \mathfrak{WR}^2 \iff Z \text{ of type } \mathfrak{WR}^1$$

⋆ If for the normalization $\nu : \overline{Z} \to Z$, $\overline{Z}_s := \nu^{-1}(\pi^{-1}(s)))$ is reduced with a singular locus of codimension at least two in the fiber for all s, then $\bar{\pi}$ is of type $\mathfrak{WR}^2$. This is because the normalization morphism of Z_s necessarily factors through $\overline{Z}_s$ (cf. [G], **Lemma 2.48, p. 343**), which is then of type $\mathfrak{WR}^1$ and thus $\mathfrak{WR}^2$ by hypothesis on the codimension of the singular locus.

We draw the reader's attention to the fact that, in general, the morphism $\bar{\pi}$ is not reduced but remains open and equidimensional over any normal base (in fact, it is geometrically flat, meaning it describes an analytic family of cycles).

(1) $\pi : Z \to S$ reduced of type $\mathfrak{WR}^1$ with smooth base of dimension 1:

(a) Case Z normal: After localization if necessary, we essentially reduce to the case where S is the unit disk $\mathbb{D}$, Z_s is of type $\mathfrak{WR}^2$ for all $s \neq 0$, and Z_0 is of type $\mathfrak{WR}^1$. Theorem 0.2 ensures that Z is of type $\mathfrak{WR}^2$ on $Z \setminus Z_0$, and we may assume all fibers are smooth outside the central fiber.

Since Z is normal, the intersection $\mathrm{Sing}(Z) \cap Z_0$ has empty interior in Z_0 (or equivalently, $\mathrm{Reg}(Z) \cap Z_0$ is a dense open subset of Z_0). Given our hypotheses, we only need to study what happens near the points of $\mathrm{Sing}(Z) \cap \mathrm{Sing}(Z_0)$, since there are no problem outside Z_0 and on $\mathrm{Sing}(Z) \cap \mathrm{Reg}(Z_0)$, we are back in the situation of Theorem 0.2. We may therefore assume that Z is irreducible and $\mathrm{Sing}(Z) = \mathrm{Sing}(Z_0)$ (which is of codimension at least two in Z).

The property for Z to be of type $\mathfrak{WR}^1$ is local in nature, as it amounts to showing that the sheaf $\mathcal{L}_Z^m$ has depth at least two over Z, which can be checked stalk by stalk. Since $\pi : Z \to \mathbb{D}$ is open with reduced fibers and hence flat, it allows lifting, at the level of local rings, any non-zero divisor to a non-zero

---

[6]Moreover, since the fibers are reduced, this flat morphism is of type S_1. The set of points of Z that are Cohen-Macaulay in the fibers $\mathrm{CM}(\pi)$ is a dense open subset of Z whose complement is a closed set with empty interior of codimension at least two fiberwise, as shown in Lemmas **(5.2.2)** and **(5.2.5)** of [A.L].

divisor[7] thanks to ([Fi], **Lemma, p.157**). Moreover, its germ π_z, at each point z of Z, defines a non-zero divisor in $O_{Z,z}$. Thus, every point of Z has an open neighborhood on which we have a representative of this germ, which we denote f, inducing the short exact sequence

$$0 \longrightarrow O_Z \xrightarrow{\ f\ } O_Z \longrightarrow i_{0*}O_{Z_0} \longrightarrow 0$$

to which we apply the functors $\mathcal{H}om(-, \omega_Z^m)$ and $\mathcal{H}om(-, \mathcal{L}_Z^m)$ to obtain the commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_Z^m & \xrightarrow{\ f\ } & \mathcal{L}_Z^m & \longrightarrow & \mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m) & \longrightarrow & 0 \\
& & \big\updownarrow & & \big\updownarrow & & \big\downarrow & & \\
0 & \longrightarrow & \omega_Z^m & \xrightarrow{\ f\ } & \omega_Z^m & \longrightarrow & \mathcal{E}xt^1(i_{0*}O_{Z_0}, \omega_Z^m) & \longrightarrow & 0
\end{array}
$$

where the vertical arrows are generically bijective and naturally injective for the first two. We will show that the same holds for the last one by verifying the absence of O_{Z_0}-torsion for the sheaf $\mathcal{E}xt^1(i_*O_{Z_0}, \mathcal{L}_Z^m)$.

To see this, consider the diagram

$$
\begin{array}{ccc}
\widetilde{Z}_0' \overset{\tilde{i'}_0}{\underset{\sigma}{\rightrightarrows}} \widetilde{Z}_0 \overset{\tilde{i}_0}{\hookrightarrow} \widetilde{Z} \\
\varphi_0' \searrow \quad \downarrow \varphi_0 \quad \downarrow \varphi \\
Z_0 \overset{i_0}{\hookrightarrow} Z
\end{array}
$$

where $\varphi : \widetilde{Z} \to Z$ is a desingularization such that the total preimage $\widetilde{Z}_0$ is a simple normal crossing divisor and the strict preimage $\widetilde{Z}_0'$ is smooth (and a finite union of certain irreducible components of $\widetilde{Z}_0$). The horizontal arrows are embeddings, and the others are natural restrictions of φ.

Abusively denoting $\tilde{f} := \varphi^*(f)$, we have the short exact sequence

$$0 \longrightarrow O_{\widetilde{Z}} \xrightarrow{\ \tilde{f}\ } O_{\widetilde{Z}} \longrightarrow \tilde{i}_{0*}O_{\widetilde{Z}_0} \longrightarrow 0$$

to which we apply the functor $\mathcal{H}om(-, \Omega_{\widetilde{Z}}^m)$ to obtain the short exact sequence

$$0 \longrightarrow \Omega_{\widetilde{Z}}^m \xrightarrow{\ \tilde{f}\ } \Omega_{\widetilde{Z}}^m \longrightarrow \tilde{i}_{0*}\omega_{\widetilde{Z}_0}^{m-1} \longrightarrow 0$$

since $\tilde{i}_{0*}\omega_{\widetilde{Z}_0}^{m-1} \simeq \mathcal{E}xt^1(\tilde{i}_{0*}O_{\widetilde{Z}_0}, \Omega_{\widetilde{Z}}^m)$. Hence, thanks to [GR70], the exact sequence

$$0 \longrightarrow \mathcal{L}_Z^m \xrightarrow{\ f\ } \mathcal{L}_Z^m \longrightarrow i_{0*}\varphi_{0*}(\omega_{\widetilde{Z}_0}^{m-1}) \longrightarrow 0 \ .$$

From this, we deduce the isomorphisms

$$\tilde{i}_{0*}\omega_{\widetilde{Z}_0}^{m-1} \simeq \Omega_{\widetilde{Z}}^m / \tilde{f}.\Omega_{\widetilde{Z}}^m, \quad \varphi_*(\tilde{i}_{0*}\omega_{\widetilde{Z}_0}^{m-1}) \simeq \mathcal{L}_Z^m / f.\mathcal{L}_Z^m \simeq \mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)$$

which incidentally shows that the sheaves $\mathcal{L}_Z^m / f.\mathcal{L}_Z^m$ and $\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)$ are torsion-free over Z_0, being the proper direct image of a torsion-free sheaf. It immediately follows that the last vertical arrow is injective due to its generic bijectivity and the absence of torsion.

Moreover, we have two natural morphisms

$$i_{0*}\mathcal{L}_{Z_0}^{m-1} \to \mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)$$

---

[7]Even in a slightly more general setting, it is always possible, according to ([G], **Lemma 2.48, p.343**), to find a germ of holomorphic function on Z that is a non-zero divisor.

$$\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m) \to i_{0*}\mathcal{L}_{Z_0}^{m-1}$$

• The first one, already mentioned in (**§5.1 (i), p.27**), is injective and can be deduced, in this new point of view, from the direct image or trace at the level of dualizing complexes[8] $\sigma_*\mathcal{D}_{\widetilde{Z}_0'}^{\bullet} \to \mathcal{D}_{\widetilde{Z}_0}^{\bullet}$ whose degree $-(m-1)$ cohomology yields the morphism of dualizing sheaves (note that $\widetilde{Z}_0'$ and $\widetilde{Z}_0$ are Gorenstein)[9] $\sigma_*\omega_{\widetilde{Z}_0'}^{m-1} \to \omega_{\widetilde{Z}_0}^{m-1}$ to which we apply the functor $i_{0*} \circ \varphi_{0*}$ to obtain

$$
\begin{array}{ccc}
i_{0*}(\varphi_{0*}'\omega_{\widetilde{Z}_0'}^{m-1}) & \longrightarrow & i_{0*}(\varphi_{0*}(\omega_{\widetilde{Z}_0}^{m-1})) \\
\Vert\wr & & \Vert\wr \\
i_{0*}\mathcal{L}_{Z_0}^{m-1} & \longrightarrow & \mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)
\end{array}
$$

Since Z is normal and Z_0 has codimension 1, this arrow is a generic isomorphism between torsion-free sheaves on Z_0, hence injective.

• To prove the existence of the second morphism, we first observe that for a finite morphism $f : X \to Y$ of Gorenstein spaces of the same dimension m, there exists a natural morphism $f_*O_X \to O_Y$ induced by the usual trace morphism defined at the level of dualizing complexes $f_*\mathcal{D}_X^\bullet \to \mathcal{D}_Y^\bullet$; these dualizing complexes have their only non-zero homology in degree $-m$, which is a locally free dualizing sheaf (of rank 1). In our particular situation, we can invoke more elementary arguments to define the morphism $\sigma_*O_{\widetilde{Z}_0'} \to O_{\widetilde{Z}_0}$ by reducing to the case where $\widetilde{Z}_0'$ is an irreducible component of $\widetilde{Z}_0$, in which case the assertion is trivial. Indeed, if $\mathcal{I}$ is the defining ideal of $\widetilde{Z}_0'$ in $\widetilde{Z}_0$ such that $\sigma_*O_{\widetilde{Z}_0'} = O_{\widetilde{Z}_0}/\mathcal{I}$, we naturally have the short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow O_{\widetilde{Z}_0} \underset{\beta}{\overset{\overset{\alpha}{\frown}}{\longrightarrow}} O_{\widetilde{Z}_0}/\mathcal{I} \longrightarrow 0$$

and the morphism $\alpha \circ \beta \in \mathcal{H}om(O_{\widetilde{Z}_0}, O_{\widetilde{Z}_0})$ is annihilated by $\mathcal{I}$.

Then, primarily using duality for a proper morphism, the adjunction formula, and the previous remark, we easily highlight the sequence of natural compositions:

$$
\begin{array}{ccc}
\mathbb{R}\mathcal{H}om(i_{0*}O_{\widetilde{Z}_0}, \mathcal{L}_Z^m) \overset{\sim}{\frown} \mathbb{R}\varphi_*\mathbb{R}\mathcal{H}om(\mathbb{L}\varphi^*(i_{0*}O_{Z_0}), \mathcal{D}_{\widetilde{Z}}^\bullet[-m]) \overset{\sim}{\frown} \mathbb{R}\varphi_*\tilde{i}_{0*}\mathbb{R}\mathcal{H}om(\mathbb{L}\varphi_0^*(O_{Z_0}), \mathcal{D}_{\widetilde{Z}_0}^\bullet[-m]) \\
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow u \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i_{0*}\mathbb{R}\varphi_{0*}\mathbb{R}\mathcal{H}om(\mathbb{L}\varphi_0^*(O_{Z_0}), \sigma_*\mathcal{D}_{\widetilde{Z}_0'}^\bullet[-m]) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Vert\wr \\
i_{0*}\mathcal{L}_{Z_0}^{m-1}[-1] \overset{\sim}{\frown} i_{0*}\mathbb{R}\mathcal{H}om(O_{Z_0}, \mathbb{R}\varphi_{0*}'\mathcal{D}_{\widetilde{Z}_0'}^\bullet[-m]) \overset{\sim}{\frown} i_{0*}\mathbb{R}\varphi_{0*}\sigma_*\mathbb{R}\mathcal{H}om(\mathbb{L}\varphi_0'^*(O_{Z_0}), \mathcal{D}_{\widetilde{Z}_0'}^\bullet[-m])
\end{array}
$$

The vertical arrow u is induced by the morphism $\sigma_*O_{\widetilde{Z}_0'} \to O_{\widetilde{Z}_0}$ since it suffices to write

$$\mathcal{D}_{\widetilde{Z}_0}^\bullet = \mathbb{R}\mathcal{H}om(O_{\widetilde{Z}_0}, \mathcal{D}_{\widetilde{Z}_0}^\bullet) \to \mathbb{R}\mathcal{H}om(\sigma_*O_{\widetilde{Z}_0'}, \mathcal{D}_{\widetilde{Z}_0}^\bullet) = \sigma_*\mathcal{D}_{\widetilde{Z}_0'}^\bullet$$

We thus define, in particular, a morphism

$$\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m) \to i_{0*}\mathcal{L}_{Z_0}^{m-1}$$

---

[8]More simply, one can use the pull back $\mathcal{L}_{Z_0}^{m-1} \to \varphi_{0*}\mathcal{L}_{\widetilde{Z}_0}^{m-1}$ which is composed with the natural morphism $\mathcal{L}_{\widetilde{Z}_0}^{m-1} \to \omega_{\widetilde{Z}_0}^{m-1}$ and apply the functor i_{0*}.

[9]Note that since $\widetilde{Z}_0$ is a simple normal crossing divisor, its normalization $\nu : \overline{\widetilde{Z}}_0 \to \widetilde{Z}_0$ is smooth. Consequently, the composition $\overline{\widetilde{Z}}_0 \to \widetilde{Z}_0 \to Z_0$ is a resolution of Z_0, and thus $\varphi_*\mathcal{L}_{\widetilde{Z}_0}^{m-1} = \mathcal{L}_{Z_0}^{m-1}$.

Note that these morphisms are constructed so that the composition

$$i_{0*}\mathcal{L}_{Z_0}^{m-1} \to \mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m) \to i_{0*}\mathcal{L}_{Z_0}^{m-1}$$

is the natural bijection, thus forcing the first arrow to be injective and the second to be surjective. This suggests that the sheaf $\mathcal{L}_Z^m$ could be a direct factor of $\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)$.

However, since the singular locus of Z has codimension at least two, these are generic isomorphisms, and since the sheaf $\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)$ is torsion-free on Z_0, the second morphism is necessarily injective and therefore bijective (since it is already injective in the other direction). The initial short exact sequence can then be written as

$$0 \longrightarrow \mathcal{L}_Z^m \overset{f}{\longrightarrow} \mathcal{L}_Z^m \longrightarrow i_{0*}\mathcal{L}_{Z_0}^{m-1} \longrightarrow 0$$

from which we can deduce, thanks to our assumptions, that the sheaf $\mathcal{L}_Z^m$ must have depth at least two.

For simplicity, we will assume Z and Z_0 are Stein since the problem is local and allows us to work on Stein open subsets of Z whose restrictions are also Stein on Z_0. Then, using the characterization of depth via cohomology with compact support ([Ba1], ([BS] **I, §3, Theorem 3.6, p.77**)), we obtain, since $\mathcal{L}_{Z_0}^{m-1}$ has depth at least two by hypothesis, the isomorphism induced by multiplication by f:

$$\mathrm{H}_c^1(Z, \mathcal{L}_Z^m) \simeq \mathrm{H}_c^1(Z, \mathcal{L}_Z^m)$$

But since

$$\mathrm{H}_c^1(Z, \mathcal{L}_Z^m) \simeq \mathrm{H}_c^1(\widetilde{Z}, \Omega_{\widetilde{Z}}^m)$$

Serre duality gives us

$$\mathrm{H}_c^1(\widetilde{Z}, \Omega_{\widetilde{Z}}^m) \simeq \left(\mathrm{H}^{m-1}(\widetilde{Z}, O_{\widetilde{Z}})\right)'$$

and the Leray spectral sequence, in turn, provides the isomorphism

$$\mathrm{H}^{m-1}(\widetilde{Z}, O_{\widetilde{Z}}) \simeq \Gamma(Z, \mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}}),$$

all these isomorphisms being topological. We thus reduce to an isomorphism between coherent sheaves (of finite type) induced by multiplication by a certain function:

$$\mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}} \simeq \mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}}$$

which, by virtue of Nakayama's lemma[10], implies that

$$\mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}} = 0$$

or equivalently,

$$\mathrm{H}_c^1(Z, \mathcal{L}_Z^m) = 0$$

that is, $\mathcal{L}_Z^m$ has depth at least two, and thus Z is of type $\mathfrak{W}\mathfrak{R}^1$

(b) Case where Z is pure-dimensional and not necessarily normal. Consider the normalization $\nu : \overline{Z} \to Z$ and the commutative diagram

---

[10]We use the variant stating that if $\mathcal{A}$ is a commutative ring, $\mathcal{I}$ an ideal of $\mathcal{A}$ contained in its Jacobson radical, and $\mathcal{M}$ a finitely generated $\mathcal{A}$-module, then

$$\mathcal{I}.\mathcal{M} = \mathcal{M} \implies \mathcal{M} = 0$$

As usual, we denote by $\text{NN}(Z)$ the non-normal locus of Z and

$$\text{NN}(\pi) := \{z \in Z : z \text{ is not normal in } \pi^{-1}(\pi(z))\}$$

that of π, which is an analytic subspace of Z ([Fi], **Prop.3.22,p.160**). Thanks to the flatness of π, we have the relations

$$\text{NN}(\pi) \cap Z_s = \text{NN}(Z_s)$$

and if S is normal (which is our case)

$$\text{NN}(Z) \subset \text{NN}(\pi), \ \text{NN}(\bar{\pi}) \cap \overline{Z}_s \subset v^{-1}(\text{NN}(Z_s))$$

We may assume that $\bar{\pi}$ is a generic simultaneous normalization, meaning that $\bar{\pi}^{-1}(s)$ is the normalization of the fiber $\pi^{-1}(s)$ for $s \neq 0$. If $\overline{Z}_0 := \bar{\pi}^{-1}(0) = v^{-1}(Z_0)$ is the central fiber, the restriction $v_0 : \overline{Z}_0 \to Z_0$ of the finite modification v is an analytic isomorphism from $\overline{Z}_0 \setminus v^{-1}(\text{NN}(Z_0))$ to $Z_0 \setminus \text{NN}(Z_0)$, but it is not a modification if $\overline{Z}_0$ is not reduced. If this fiber is reduced, then $\overline{Z}_0$ and Z_0 have the same normalization (cf.[G], **Prop2.49, p.344**), $\bar{\pi}$ is obviously flat, and we know how to conclude (cf.the third remark in **(1)**)[11].

Again, point **(ii)** of ([G], **Prop2.49, p.344**) allows us to see, knowing that Z is of pure dimension and that the dimension formula still holds for $\bar{\pi}$ which is geometrically flat since equidimensional over a normal base (in particular open), that $v^{-1}(\text{NN}(Z_0))$ has empty interior in $\overline{Z}_0$. It follows that v_0 is a finite modification and that, consequently, $\text{depth}(\mathcal{L}_{\overline{Z}_0}^{m-1}) = \text{depth}(\mathcal{L}_{Z_0}^{m-1}) \geq 2$, meaning $\overline{Z}_0$ is of type $\mathfrak{WR}^1$.

We then deduce from **(i)** that $\overline{Z}$ is of type $\mathfrak{WR}^2$ and, consequently, Z is of type $\mathfrak{WR}^1$

(2) $\pi : Z \to S$ **reduced of type** $\mathfrak{WR}^1$ **with** S **smooth of dimension** $r > 1$ **and the general case:**

Recall that the morphism $\pi : Z \to S$ is flat since it is open with reduced fibers over a smooth base. We proceed by induction on the dimension r of S as in the proof of (Theorem 0.2, **(a) ii), p.29**), assuming the result holds for any base of dimension strictly less than r. Using the notations of this paragraph, we have the Cartesian diagram of base change

in which S' is smooth of dimension 1, S" appropriately chosen of dimension $r - 1$. We see that the fiber over $\{0\}$ of ψ is given by the analytic space $Z \times_{S"} \{0\}$ analytically isomorphic to $Z' := Z \times_S S'$ which has singularities of type $\mathfrak{WR}^1$ since the morphism π' is flat with reduced fibers over a smooth base of dimension 1. We then conclude by applying the induction hypothesis to ψ to deduce that Z is of type $\mathfrak{WR}^1$.

The case of singular base is deduced from the smooth base by applying Proposition 3.7.

6.2. The case where π is universally n-equidimensional whose fibers satisfy the property (P). In this case, some fibers are not reduced.

• An intermediate step toward the case of a morphism with non-reduced fibers is suggested by the case of a flat morphism whose generic fiber is reduced and the other fibers generically reduced. In this situation and given our reductions (given at the beginning of the paragraph), we may assume our source Z reduced by virtue of ([C.T-Lê], **Theorem 2.5, p.645**) and proceed as above without significant

---

[11]If $\overline{Z}_0$ is generically reduced and if the underlying reduced space is normal, then, by a classical result ([H1], [E.G.A.4]§5.12.3), $\overline{Z}_0$ is necessarily normal and $\bar{\pi}$ flat, which means that π is equinormalizable.

modifications. If the morphism π is only universally n-equidimensional generically reduced on reduce base with all non reduced fibers generically reduced, the local flattening theorem (see **§5.2)b),p.30**) allows us to reduce to the case of a flat morphism.

In the sequel, we adopt a method that avoids the use of local flattening. First, let us recall that the non flat locus $\text{NFlat}(\pi)$, the non regular points, $\text{Sing}(\pi) := \text{NFree}(\Omega^1_{Z/S}) \cup \text{NFlat}(\pi)$ (union of the non-free locus of $\Omega^1_{Z/S}$ and the non-flat locus $\text{NFlat}(\pi)$), the non-normal locus $\text{NN}(\pi)$ and the non-reduced locus $\text{N}(\text{Red}(\pi))$ are analytically closed (cf.[Fri], [G], **Theorem 1.100, p.98**; **Theorem 1.115, p.107**). Furthermore, if π is flat and Z reduced, the complementary set $\text{Reg}(\pi)$ [12] (resp. $\text{Free}(\Omega^1_{Z/S})$, $\text{Flat}(\pi)$, $\text{N}(\pi)$ and $\text{Red}(\pi)$) is a dense open subset (in general, only locally constructible). The reader can compare with ([Fi], **Prop.3.22, p.160**) in which we have only the local analytically constructibility of $\text{S}(\pi) := \text{Sing}(\pi) \setminus \text{NFlat}(\pi)$.

We begin to precise the behavior of the condition

$$(P) \quad \overline{\pi^{-1}(s) \cap \overset{\circ}{\text{Sing}(\pi)}} = \emptyset \text{ in } \pi^{-1}(s), \ \forall s \in S$$

by certain base change.

Remark that for any commutative diagram

$$
\begin{array}{ccc}
\overline{Z} & \xrightarrow{\theta} & Z \\
\bar{\pi} \downarrow & & \downarrow \pi \\
\overline{S} & \xrightarrow{\varphi} & S
\end{array}
$$

induced by an arbitrary base change φ, we have

$$\text{Sing}(\tilde{\pi}) \subset \theta^{-1}(\text{Sing}(\pi))$$

For this, it suffise to have in mind that $\theta^*(\Omega^1_{Z/S}) \simeq \Omega^1_{\tilde{Z}/\tilde{S}}$, the flatness and the local freeness are preserved by base change.

• **stability of** (P) **under any base change induced by a proper modification.**

Indeed, consider the base change diagram:

relative to a proper modification with singular locus as center $\nu : \overline{S} \to S$ and the inclusion $\text{Reg}(\pi) \hookrightarrow Z$. Since smooth morphisms are stable under arbitrary base changes, $\bar{\pi}'$ is smooth because π' is smooth, and we have $\theta^{-1}(\text{Reg}(\pi)) \subset \text{Reg}(\bar{\pi})$. Then, since π is universally n-equidimensional, its restriction to the dense open subset $\text{Reg}(\pi)$ remains a universally n-equidimensional morphism and even flat by definition. It follows that $S' := \pi(\text{Reg}(\pi))$ is a dense open subset of S over which the fibers are smooth

---

[12]Instead of $\text{Reg}(\pi)$, one can take the Cohen-Macaulay locus $\text{CM}(\pi)$, which is a dense open subset of Z by virtue of Frisch's theorem ([Fri]) and the fact that $\text{Reg}(\pi) \subset \text{CM}(\pi)$.

and hence itself smooth. Consequently, the restriction ν' (resp. θ') of the modification ν (resp. θ) to $\nu^{-1}(S')$ (resp. $\theta^{-1}(\mathrm{Reg}(\pi))$) is an analytic isomorphism.

Now, since θ is proper, the inclusion $\mathrm{Sing}(\bar{\pi}) \cap \overline{Z}_{\bar{s}} \subset \theta^{-1}(\mathrm{Sing}(\pi)) \cap \overline{Z}_{\bar{s}}$ yields the inclusions:

$$\theta(\mathrm{Sing}(\bar{\pi}) \cap \overline{Z}_{\bar{s}}) \subseteq \mathrm{Sing}(\pi) \cap \theta(\overline{Z}_{\bar{s}}) = \mathrm{Sing}(\pi) \cap Z_s \subsetneq Z_s$$

which shows that (P) is satisfied for $\bar{\pi}$ since $\overline{Z}_{\bar{s}}$ is analytically isomorphic to Z_s and thus has the same irreducible components (which prevents $\mathrm{Sing}(\bar{\pi})$ from having a common irreducible component with $\overline{Z}_{\bar{s}}$).

More generally, if F is a non-π-saturated closed subset—meaning that $F \cap \pi^{-1}(\pi(x))$ has no irreducible component in common with $\pi^{-1}(\pi(x))$ for all $x \in F$—then the closed subset $\theta^{-1}(F)$ is also non-$\bar{\pi}$-saturated. This is easily understood knowing that:

$$\theta^{-1}(F) \cap \overline{Z}_{\bar{s}} \simeq (F \cap Z_s) \times_S \overline{S}$$

- **stability of (P) under any flat base change.**

With the previous base change diagram and by considering φ flat (and then θ is flat too), it is easy to see that $\theta^*(\mathrm{Flat}(\pi)) = \mathrm{Flat}(\tilde{\pi})$ and $\theta^*(\mathrm{Free}(\Omega^1_{Z/S}) = \mathrm{Free}(\Omega^1_{\tilde{Z}/\tilde{S}})$ so that

$$\mathrm{Sing}(\tilde{\pi}) = \theta^{-1}(\mathrm{Sing}(\pi))$$

- **stability of (P) under any base change between smooth bases.**

We can assume that S is a smooth open subset and S' a smooth subvariety equipped with the embedding $S' \hookrightarrow S$, and possibly after restricting the data, we can assume S equipped with a system of parameters $(s_1, s_2, \cdots, s_r)$ and S' a hypersurface given by the vanishing of a certain coordinate and then see that this condition is obviously satisfied over S'.

Let us point out that (P) implies that the two following incidence conditions are fulfilled

$$(P') \quad \overset{\circ}{\overline{\pi^{-1}(s) \cap \mathrm{NN}(\pi)}} = \emptyset \text{ in } \pi^{-1}(s), \ \forall\, s \in S$$

$$(P'') \quad \overset{\circ}{\overline{\pi^{-1}(s) \cap \mathrm{NN}(Z)}} = \emptyset \text{ in } \pi^{-1}(s), \ \forall\, s \in S$$

Since Z and S are reduced and of pure dimension, these incidence relations are generically satisfied on S, meaning that there exists a dense open subset of S over which all fibers satisfy (P), (P') or (P'') and it is easy to verify that $(P) \implies (P') \implies (P'')$.

Assume S smooth. Since every regular point z for π is, a fortiori, a point of flatness and normality on the fiber $\pi^{-1}(\pi(z))$, we have:

$$\mathrm{Reg}(\pi) \subset \mathrm{N}(\pi)$$

On the other hand, if z is a normal point of the fiber $Z_{\pi(z)}$, then it is necessarily normal in Z because, at this point, the fiber is reduced which implies the flatness of the morphism in z (because open and reduced in z) and the normality of Z at z. Thus, $\mathrm{N}(\pi) \subset \mathrm{N}(Z)$ and, therefore:

$$\mathrm{NN}(Z) \cap Z_s \subset \mathrm{NN}(\pi) \cap Z_s \subset \mathrm{Sing}(\pi) \cap Z_s$$

It is interesting to note that condition (P'') is preserved if the base change is given by the normalization morphism of S.

Indeed, let us consider the base change diagram

$$\begin{array}{ccc} \overline{Z} & \xrightarrow{\ \theta\ } & Z \\ {\scriptstyle \bar{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \overline{S} & \xrightarrow[\ \nu\ ]{} & S \end{array}$$

induced by the normalization $\nu : \overline{S} \to S$ and where π and $\bar{\pi}$ are universally n-equidimensional and θ is finite modification.

Then the restricted morphism $\theta' : \theta^{-1}(\mathrm{N}(Z)) \to \mathrm{N}(Z)$ is a finite modification too. Furthermore, $\theta^{-1}(\mathrm{N}(Z))$ is necessarily normal which follows from the following fact:

Let $f : X \to Y$ finite modification of normal complex space Y, then X is normal too.

For see that, we can consider the exact sequence

$$0 \to O_X \to \mathcal{L}_X^0 \to \mathcal{K} \to 0$$

and remark that f is necessarily geometrically flat because equidimensionnal over normal base (and especially open morphism!). By finiteness of f, we deduce, the short exact sequence

$$0 \to f_*O_X \to f_*\mathcal{L}_X^0 \to f_*\mathcal{K} \to 0$$

But f is a proper modification so that $f_*\mathcal{L}_X^0 = \mathcal{L}_Y^0$ and then by normality of Y, $f_*\mathcal{L}_X^0 = O_Y$. Thus, we have

$$0 \to f_*O_X \to O_Y \to f_*\mathcal{K} \to 0$$

Finally, we use the fact that f induces a trace map that allows us to see O_Y as a direct factor of f_*O_X (and then an injection of the first sheaf in the second) which implies, in view of the exact sequence above, that $f_*\mathcal{K} = 0$ and then $\mathcal{K} = 0$ (f is finite).

From this result, we deduce that $\theta^{-1}(\mathrm{N}(Z)) \subset \mathrm{N}(\overline{Z})$ or equivalently $\mathrm{NN}(\overline{Z}) \subset \theta^{-1}(\mathrm{NN}(Z))$. We conclude as above for $\mathrm{Sing}(\pi)$.

This remark allows us to assume S normal if we want.

Thanks to Proposition 3.8 and what has just been said above, we proceed exactly as in **§5.1 p.27** and **§6.1 p.34**, assuming S smooth and begin by treating:

(a) the case of a 1-dimensional base and Z normal.

We can assume that we have only one central fiber Z_0 and use the same notations and considerations as in **§5.1** to exhibit the short exact sequence

$$0 \longrightarrow \tilde{f}.O_Z \xrightarrow{\bullet \tilde{f}} O_Z \longrightarrow i_{0*}O_{Z_0} \longrightarrow 0$$

to which we apply the functor $\mathcal{H}om(-, \mathcal{L}_Z^m)$ giving us the exact sequence

$$0 \longrightarrow \mathcal{L}_Z^m \longrightarrow \mathcal{L}_{Z,\tilde{f}}^m \longrightarrow \mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m) \longrightarrow 0$$

with $\mathcal{L}_{Z,\tilde{f}}^m := \mathcal{H}om(\tilde{f}.O_Z, \mathcal{L}_Z^m)$.

Just as in (**§6.1, (1).a; p.35**), we can easily see that the sheaf $\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)$ is torsion-free on Z_0 since isomorphic to the torsion-free sheaf $i_{0*}(\varphi_{0*}(\omega_{Z_0}^{m-1}))$.

While the morphism

$$i_{0*}\mathcal{L}_{Z_0}^{m-1} \to \mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)$$

is always injective, the injectivity of

$$\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m) \to i_{0*}\mathcal{L}_{Z_0}^{m-1}$$

requires the existence of a dense open subset of Z whose trace on Z_0 is still dense and on which this morphism is bijective.

Fortunately, this is given by the hypothesis since $\mathrm{Sing}(Z) \cap Z_0$ is assumed to have empty interior in Z_0. We can then work generically on Z and Z_0 and establish the desired injectivity and consequently the isomorphism

$$i_{0*}\mathcal{L}_{Z_0}^{m-1} \simeq \mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m).$$

The sheaf $\mathcal{L}_{Z_0}^{m-1}$ being, by hypothesis, of depth at least two, the above short exact sequence and the vanishing $\mathrm{H}_c^0(Z, \mathcal{L}_{Z_0}^{m-1}) \simeq \mathrm{H}_c^1(Z, \mathcal{L}_{Z_0}^{m-1}) = 0$ give us the isomorphism

$$\mathrm{H}_c^1(Z, \mathcal{L}_Z^m) \simeq \mathrm{H}_c^1(Z, \mathcal{L}_{Z,\hat{f}}^m),$$

To show that this necessarily implies $\mathrm{H}_c^1(Z, \mathcal{L}_Z^m) = 0$, we proceed as in the case of the reduced morphism where we had obtained

$$\mathrm{H}_c^1(Z, \mathcal{L}_Z^m) \simeq \mathrm{H}_c^1(\widetilde{Z}, \Omega_{\widetilde{Z}}^m) \simeq \left(\mathrm{H}^{m-1}(\widetilde{Z}, O_{\widetilde{Z}})\right)' \simeq \left(\Gamma(Z, \mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}})\right)'$$

Then, denoting by $\mathcal{I}$ (resp. $\tilde{\mathcal{I}}$) the ideal generated by f (resp. $\varphi^*(f)$), we have

$$\mathrm{H}_c^1(Z, \mathcal{L}_{Z,\hat{f}}^m) = \mathrm{H}_c^1(Z, \mathcal{H}om(\mathcal{I}O_Z, \mathcal{L}_Z^m)) \simeq \mathrm{H}_c^1(Z, \varphi_*\mathcal{H}om(\tilde{\mathcal{I}}O_{\widetilde{Z}}, \Omega_{\widetilde{Z}}^m))$$

Since

$$\mathbb{R}^i\varphi_*((\tilde{\mathcal{I}}O_{\widetilde{Z}})^* \otimes \Omega_{\widetilde{Z}}^m) = 0, \ \forall\, i \neq 0$$

we obtain

$$\mathrm{H}_c^1(Z, \mathcal{L}_{Z,\hat{f}}^m) \simeq \mathrm{H}_c^1(\widetilde{Z}, \tilde{\mathcal{I}}O_{\widetilde{Z}})^* \otimes \Omega_{\widetilde{Z}}^m) \simeq \left(\mathrm{H}^{m-1}(\widetilde{Z}, \tilde{\mathcal{I}}O_{\widetilde{Z}})\right)' \simeq \left(\Gamma(Z, \mathcal{I} \otimes \mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}})\right)'.$$

The initial isomorphism is therefore equivalent to the isomorphism between coherent sheaves (of finite type) induced by multiplication by $\mathcal{I}$

$$\mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}} \simeq \mathcal{I} \otimes \mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}}.$$

We then conclude by applying Nakayama's lemma as used above, concluding with the vanishing of $\mathbb{R}^{m-1}\varphi_* O_{\widetilde{Z}}$ and thus that of $\mathrm{H}_c^1(Z, \mathcal{L}_Z^m)$.

(b) The case of a base of dimension 1 with Z reduced and pure-dimensional.

We can revisit the arguments developed in (**§6.1, 1.b), p.37**) by again considering the commutative diagram

where ν is the normalization morphism. We note that if z is a normal point of the fiber $Z_{\pi(z)}$, then it is necessarily normal in Z since, at this point, the fiber is reduced and, consequently, the morphism is flat at z. In this case, Z is normal at z. Thus, $\mathrm{NN}(Z) \cap Z_s \subset \mathrm{NN}(Z_s)$ and, therefore, the finite morphism induced by ν on $\overline{Z_s}$ is a biholomorphism from $\overline{Z_s} \setminus \nu^{-1}(\mathrm{NN}(Z) \cap Z_s)$ to $Z_s \setminus \mathrm{NN}(Z) \cap Z_s$. Since, by hypothesis, $\mathrm{NN}(Z) \cap Z_s$ has empty interior in Z_s, we conclude in the same way as in (**§6.1, (b), p.37**) by showing that $\nu_s : \overline{Z_s} \to Z_s$ is a modification. We then deduce, thanks to case **(a)**, that Z is of type $\mathfrak{WR}^1$.

(c) The case of a general reduced base. As in (**§6.1, (2), p.37**), we extend to the case of a smooth base of dimension greater than 1 by noting that the condition (P) is stable by base change between smooth bases. Indeed if S is a smooth open subset and S' a smooth subvariety equipped with the embedding $S' \hookrightarrow S$, and possibly after restricting the data, we can assume S equipped with a system of parameters $(s_1, s_2, \cdots, s_r)$ and S' a hypersurface given by the vanishing of a certain coordinate and then see that this condition is obviously satisfied over S'.

The case of a reduced base of type $\mathfrak{WR}^1$ is deduced from the smooth base thanks to the Proposition 3.8

We easily deduce the following consequence in the style of [Ste]:

Corollary 6.1. *Let $\pi : Z \to \mathbb{D}$ be a flat and reduced morphism over the complex unit disk whose central fiber Z_0 is of type $\mathfrak{WR}^1$ and whose generic fiber Z_t ($t \neq 0$) is of type $\mathfrak{WR}^2$, then Z is of type $\mathfrak{WR}^2$.*

Proof. According to Theorem 0.2, $Z \setminus Z_0$ is of type $\mathfrak{WR}^2$ and, according to Theorem 0.3, Z is of type $\mathfrak{WR}^1$. Since π is flat, we have, for all $t \in \mathbb{D}$, $\mathrm{Sing}(Z) \cap Z_t = \mathrm{Sing}(Z_t)$. Of course, we can assume $\Sigma := \mathrm{Sing}(Z)$ irreducible of pure codimension 1 because otherwise there is nothing to prove since Z is already of type $\mathfrak{WR}^1$.

Since $\dim(\mathrm{Sing}(Z_0)) \leq m - 2$ and the extension problem occurs precisely in the neighborhood of points of $\mathrm{Sing}(Z_0)$, the result follows due to the depth of at least two of the two sheaves $\mathcal{L}_Z^m$ and ω_Z^m. $\square$

7. Transfer of properties $\mathfrak{WR}^2$ and $\mathfrak{WR}^1$ from the total space to the fibers: Theorem 0.4

Let Z and S be two reduced complex spaces of pure dimensions m and r, respectively. Let $\pi : Z \to S$ be an open morphism with fibers of pure dimension n and reduced. We assume that π admits a simultaneous resolution, and that Z and S are of type $\mathfrak{WR}^2$ (resp. $\mathfrak{WR}^1$). Let us show that π is of type $\mathfrak{WR}^2$ (resp. $\mathfrak{WR}^1$).

First, note that we may assume S is smooth. Indeed, let $\varphi : \widetilde{Z} \to Z$ be a simultaneous resolution of singularities for π, and let $\nu : \widetilde{S} \to S$ be a desingularization of S. Then, we have the commutative diagram of base change

$$
\begin{array}{ccc}
\widetilde{S} \times_S \widetilde{Z} & \xrightarrow{\ \theta\ } & \widetilde{Z} \\
{\scriptstyle \tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle \varphi} \\
\widetilde{S} \times_S Z & \xrightarrow{\ \tilde{\nu}\ } & Z \\
{\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\
\widetilde{S} & \xrightarrow{\ \nu\ } & S
\end{array}
$$

Since smooth morphisms are stable under arbitrary base change, it follows that $\tilde{\Psi}$ is a smooth morphism and, consequently, $\tilde{\varphi}$ is a simultaneous resolution for $\tilde{\pi}$. Thus, thanks to Lemme 3.3 or Lemme 3.2 and Proposition 3.8, we see that if the result holds for $\tilde{\pi}$, it will also hold for π, whose fibers are analytically isomorphic to those of $\tilde{\pi}$.

(i) Case $\mathfrak{WR}^2$: This is an immediate consequence of ([AS84], **Lemma 5.8,p.74**), which gives the existence of a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{L}_Z^m & \longrightarrow & \mathcal{L}_{Z_s}^n & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\omega_Z^m & \longrightarrow & \omega_{Z_s}^n & \longrightarrow & 0
\end{array}
$$

(ii) Case $\mathfrak{WR}^1$: We factorize φ through the normalization $\nu : \overline{Z} \to Z$ to obtain the commutative diagram

$$
\begin{array}{c}
\widetilde{Z} \\
{\scriptstyle \tilde{\varphi}}\swarrow \quad \searrow{\scriptstyle \varphi} \\
\overline{Z} \xrightarrow{\ \nu\ } Z \\
{\scriptstyle \bar{\pi}}\searrow \quad \swarrow{\scriptstyle \pi} \\
S
\end{array}
$$

The morphism π is flat because it is open with reduced fibers over a smooth base, the morphism Ψ, by hypothesis, is smooth, and the fibers $\widetilde{Z}_s$ are desingularizations of the fibers Z_s. The morphism $\bar{\pi}$ is

also open with fibers of pure dimension n because it is equidimensional over a smooth base and thus flat. Moreover, these fibers are necessarily reduced since their preimages by $\tilde{\varphi}$ are smooth and hence reduced. It follows that $\tilde{\varphi}$ is a simultaneous resolution of $\bar{\pi}$.

Then, we have

$$Z \text{ of type } \mathfrak{W}\mathfrak{R}^1 \Longleftrightarrow \overline{Z} \text{ of type } \mathfrak{W}\mathfrak{R}^2 \Longrightarrow \overline{Z}_s \text{ of type } \mathfrak{W}\mathfrak{R}^2 \Longrightarrow Z_s \text{ of type } \mathfrak{W}\mathfrak{R}^1$$

where the first equivalence is given by Proposition 2.9, the second implication is deduced from case **(i)**, and the last is due to the fact that the morphism induced by ν on each fiber $\overline{Z}_s$ is a finite modification dominated, moreover, by the normalization of Z_s (which is also that of $\overline{Z}_s$).

8. FINAL REMARKS AND SOME EXAMPLES

8.1. **About the Theorem 0.3:** **(i)** If Z_0 is an hypersurface of the Cohen Macaulay reduced complex spaces Z, the hypothesis $\text{depth}(\mathcal{L}_{Z_0}^{m-1}) := \text{depth}(\mathcal{L}_Z^{m-1}|_{Z_0}) \geq 2$ implies that $\text{depth}(\mathcal{L}_Z^{m-1}) \geq 3$ due to the well-known fact (in algebraic or analytic geometry) that if Z_0 is a Cartier divisor of a reduced analytic pure-dimensional complex space Z and $\mathcal{F}$ is an S_1-coherent sheaf on Z, then, at a point z,

$$\text{depth}(\mathcal{F}_z) \geq d + 1 \iff \text{depth}((\mathcal{F}|_{Z_0})_z) \geq d$$

Thus, for Z normal, we have

$$\mathcal{L}_Z^{m-1} \simeq \omega_Z^{m-1}$$

which, by virtue of ([KeSc], **Theorem (1.2), Theorem 4**), implies the isomorphisms

$$\mathcal{L}_Z^j \simeq \omega_Z^j, \ \forall \, j \in \{0, 1, \cdots, m-1\}$$

enforcing the inclusions [13]

$$\mathcal{L}_Z^j \subset \mathcal{H}om(\mathcal{L}_Z^{m-j}, \mathcal{L}_Z^m) \subset \mathcal{H}om(\Omega_Z^{m-j}, \mathcal{L}_Z^m) \subset \mathcal{H}om(\Omega_Z^{m-j}, \omega_Z^m) \simeq \omega_Z^j$$

to be bijective for all $j \in \{0, 1, \cdots, m-1\}$.

(ii) The arrow $\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m) \to i_{0*}\mathcal{L}_{Z_0}^{m-1}$, if there exist, must be the restriction of the residue morphism $\text{Res}_{Z_0}(\omega) : \mathcal{E}xt^1(i_{0*}O_{Z_0}, \omega_Z^m) \to i_{0*}\omega_{Z_0}^{m-1}$. For Z and Z_0 being Gorenstein, it is easy to construct the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_Z^m & \longrightarrow & \mathcal{L}_Z^m[Z_0] & \longrightarrow & \mathcal{L}_{Z_0}^{m-1} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega_Z^m & \longrightarrow & \omega_Z^m[Z_0] & \longrightarrow & \omega_{Z_0}^{m-1} & \longrightarrow & 0
\end{array}
$$

and make the calculations explicit by considering a local embedding of Z into a Stein manifold W and an embedded desingularization. By fixing a generator ω_0 of ω_Z^m that allows describing all sections of the canonical sheaf in the form $g.\omega_0$ with g an arbitrary holomorphic function on Z. Then, it is known that the sections of $\mathcal{L}_Z^m$ are exactly the forms $g.\omega_0$ with g in the adjunction ideal of Z, i.e., the holomorphic functions g such that $g.\omega_0$ is square integrable. It is clear that for any element of $\mathcal{E}xt^1(i_{0*}O_{Z_0}, \mathcal{L}_Z^m)$, the relation

$$\text{Res}_{Z_0}\left(\frac{g\omega_0}{f}\right) = g\text{Res}_{Z_0}\left(\frac{\omega_0}{f}\right)$$

shows that $\text{Res}_{Z_0}\left(\frac{g\omega_0}{f}\right)$ is a section of the sheaf $\omega_{Z_0}^{m-1}$ and as already mentioned, we can assume that the singular locus of Z is the singular locus of Z_0 and then, obviously, the restriction to Z_0 of the

---

[13]These equalities are not sufficient to enforce the equality $\mathcal{L}_Z^m \simeq \omega_Z^m$, as can be seen by revisiting the two-parameter family from ([vSS85], p. 104) given by $\mathcal{Z} := \{(x, y, z, s, t) \in \mathbb{C}^3 \times \mathbb{C}^2 : \frac{x^7}{7} + \frac{y^3}{3} + \frac{z^3}{3} - syx^5 - tyzx^4 = 0\}$, for which the arithmetic genus is zero for $s = 0$ and $t \neq 0$ and the geometric genus is non-zero for every pair (s, t); this translates into the equalities $\mathcal{L}_Z^j \simeq \omega_Z^j$ for $j = 0, 1$ and the strict inclusion $\mathcal{L}_Z^2 \subset \omega_Z^2$.

holomorphic function g still in the adjunction ideal of Z_0 which precisely means that this residue defines a section of the sheaf $\mathcal{L}_{Z_0}^{m-1}$. Since every element of $\mathcal{E}xt^1(i_{0*}\mathcal{O}_{Z_0}, \mathcal{L}_Z^m)$ can be described in this way, the conclusion follows.

8.2. **Some examples of particular morphisms.** Consider the hypersurfaces in $\mathbb{C}^4$ given respectively by $Z_1 := \{(x,y,z,t) \in \mathbb{C}^4 : x^2 + zy^2 + txz = 0\}$, $Z_2 := \{(x,y,z,t) \in \mathbb{C}^4 : x^2 + zy^3 + tz^2 = 0\}$, $Z_3 := \{(x,y,z,t) \in \mathbb{C}^4 : z^3 + xy^2 + tx^2z = 0\}$, and $Z_4 := \{(x,y,z,t) \in \mathbb{C}^4 : x^4 + y^3 + z^3 + tz^2 = 0\}$, and the morphisms $\pi_j : Z_j \to \mathbb{C}$ induced by the canonical projection $\mathbb{C}^4 \to \mathbb{C}$ sending (x,y,z,t) to t.

Explicit calculations show that these Gorenstein morphisms are, for the first three, generically normal fibers of type $\mathfrak{W}\mathfrak{R}^2$ (hence with rational singularities) with a non-normal central fiber of type $\mathfrak{W}\mathfrak{R}^1$ (Z_1 is weakly normal, Z_2 and Z_3 are not; see Remark 8.2) and a normal total space of type $\mathfrak{W}\mathfrak{R}^2$ (thus also with rational singularities since normal); for the fourth, generically normal fibers of type $\mathfrak{W}\mathfrak{R}^2$, a central fiber that is normal but not of type $\mathfrak{W}\mathfrak{R}^1$, and a total space of type $\mathfrak{W}\mathfrak{R}^2$.

One can mention that Duco van Straten shows, among other interesting results, in ([Strat], **prop.4.4.5**,p.149) that if $\pi : X \to S$ is a flat deformation of weakly normal $X_0 := \pi^{-1}(\{0\})$, equipped with a morphism $\varphi : Y \to X$ satisfying:

(i) $Y \setminus \varphi^{-1}(\mathrm{Sing}(X)) \simeq X \setminus \mathrm{Sing}(X)$,

(ii) $\varphi_* \mathcal{O}_Y \simeq \mathcal{O}_X$. Then,

$$\mathbb{R}^1 \varphi_* \mathcal{O}_Y = 0$$

This implies, in particular, that if $(X_s)_{s \in S}$ is a smoothing of weakly normal curve, the total space has necesseraly rational singularities.

One can also point out that, in this same work, Duco van Straten gives a different definition of weak rationality for surface singularities (cf.**definition 4.1.1**,p.124.): For any improvement $\varphi : Y \to X$ of weakly normal Cohen Macaulay surface (cf.**definition 1.4.2**, p.42), X is said to be weakly rational (that we noted $\mathfrak{W}\mathfrak{R}_s$) if $\mathbb{R}^1 \varphi_* \mathcal{O}_Y = 0$ and the geometric genus is defined as $\mathrm{p}_g(X, p) := \dim(\omega_X^2 / \varphi_* \omega_Y^2)_p$ and not as $\dim(\omega_X^2 / \mathcal{L}_X^2)_p$.

Then, for an improvement $\varphi : Y \to X$, X is of type $\mathfrak{W}\mathfrak{R}_s$ if and only if $\mathrm{depth}(\varphi_* \omega_Y^2) \geq 2$.

All complex spaces with smooth normalization are of $\mathfrak{W}\mathfrak{R}^1$ type but not necessarily $\mathfrak{W}\mathfrak{R}_s$ type. One can take, for example, the surface with an elliptic singularity $X := \{(x,y,z) \in \mathbb{C}^3 : z^2 - y^2(x - y) = 0\}$ which is of type $\mathfrak{W}\mathfrak{R}^1$ but not of type $\mathfrak{W}\mathfrak{R}_s$.

8.3. **Examples of geometrically flat map and analytic family of cycles.** We will give some examples of geometrically flat morphisms which are no flat and induced by the data of analytic families of cycles of dimension 1 and 2 parameterized by weakly normal (and non-weakly normal analytic spaces Remark 8.2).

As usual, $\mathrm{Sym}_0^2(\mathbb{C}^2)$ will denote the homogeneous part of degree 2 of $\mathrm{Sym}^2(\mathbb{C}^2)$, canonically identified with $\mathbb{C}^2/\{id, -id\}$, which is isomorphic to the cone $\{(x,y,z) \in \mathbb{C}^3 : xy = z^2\}$ in $\mathbb{C}^3$.

We will use:

$\star$ The functoriality of the reduction operation: Any base change diagram

$$
\begin{array}{ccc}
X_\varphi & \xrightarrow{\ \theta\ } & X \\
{\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
S' & \xrightarrow[\ \varphi\ ]{} & S
\end{array}
$$

completes into a commutative diagram by naturally including the reductions $\mathrm{Red}(X_\varphi)$ and $\mathrm{Red}(X)$ ([Fi]),

$\star$ the local description of morphisms defining analytic families of cycles as stated in ([K1], [K2], **prop.4.4p.15**) and in ([BM1], **Theorem 5.14 p.467**):

• *Let π be a universally n-equidimensional morphism (weighted by a cycle $\mathfrak{X}$). Then π is analytically geometrically flat if and only if for every S-smooth complex space Y and every finite and surjective morphism $f : X \to Y$, there exists a unique continuous O_S-linear trace morphism $\mathcal{T}^n_{f,\mathfrak{X}} : f_*\Omega^n_{X/S} \to \Omega^n_{Y/S}$ extending the usual trace $f_*f^*\Omega^n_{Y/S} \to \Omega^n_{Y/S}$, commuting with any base change over S, with localizations on X, compatible with the additivity of weights on π, and inducing (and induced by) the morphism of graded differential algebras $\mathcal{T}^{\bullet}_{f,\mathfrak{X}} : f_*\Omega^{\bullet}_{X/S} \to \Omega^{\bullet}_{Y/S}$.*

or in ([BM1], **Theorem 5.14 p.467**):

• *Let S be a reduced complex space, V a connected analytic variety, B a relatively compact polydisc in $\mathbb{C}^p$, and $F : S \times V \to \mathrm{Sym}^k(B)$ a holomorphic map. Then, the following conditions are equivalent:*

(i) *For each holomorphic form ω on $V \times B$, the trace $\mathrm{Tr}_{X_s/V}(\omega)$ is an S-relative holomorphic form on $S \times V$,*

(ii) *For each $m \in [0, k-1]$ and each index set $\mathrm{I} \subset [0, p]$, the differential form $\mathrm{Tr}_{X_s/V}(x^m dx^I)$ with values in $\mathrm{S}^m(\mathbb{C}^p)$ is an S-relative holomorphic form on $S \times V$.*

(iii) *The morphism F is isotropic (does not depend on the chosen projection).*

• Propositions *Proposition 3.7 and Proposition 3.8.*

Some calculations were performed using **Singular** [DGPS]. This allow us to see that all the complex spaces X below are non-normal and non-Cohen Macaulay. .

(1) Analytic Family of Lines: Let $S := \{(a, b, c) \in \mathbb{C}^3 : a^2 - cb^2 = 0\}$ be the weakly normal surface commonly called the "Whitney umbrella" in $\mathbb{C}^3$, whose singular locus is the line

$$\Sigma := \{(a, b, c) \in \mathbb{C}^3 : a = b = 0\}.$$

Consider the ideal in $\mathbb{C}^6$ given by $\mathcal{I} := (u^2 - ct^2, v^2 - b^2, uv - at, a^2 - cb^2)$ defining a certain subspace that we will somewhat abusively denote as $X = \{(u, v, a, b, c, t) \in \mathbb{C}^2 \times S \times \mathbb{D} : u^2 = ct^2, v^2 = b^2, uv = at, a^2 = cb^2\}$, where $\mathbb{D}$ is the unit disk. We will denote by $\pi : X \to S$ (resp. $f : X \to S \times \mathbb{D}$) the morphisms induced by the canonical projections from $S \times \mathbb{D} \times \mathbb{C}^2 \to S \times \mathbb{D} \to S$. We can view X as a fiber product in the cartesian diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \mathbb{C}^2 \\
f \downarrow & & \downarrow \varphi \\
S \times \mathbb{D} & \xrightarrow{\ g\ } & \mathbb{C}^3
\end{array}
$$

in which φ and g are defined by $\varphi(u, v) = (u^2, v^2, uv)$ and $g(a, b, c, t) = (ct^2, b^2, at)$. Since X is invariant under the involution $\sigma : (a, b, c, t, u, v) \to (a, b, c, t, -u, -v)$, it identifies, after quotient by this relation, with the graph $\mathcal{G}_F$ of the analytic map $F : S \times D \to \mathrm{Sym}^2_0(\mathbb{C}^2)$ defined by $F(a, b, c, t) = (ct^2, b^2, at)$.

For generic s, the fiber $X_s := \pi^{-1}(s)$ consists of a pair of lines in general position whose equations are:

• If $b \neq 0$:

$$
\begin{cases}
u_1 = \frac{a}{b}t, & u_2 = -u_1 \\[2mm]
v_1 = b, & v_2 = -v_1
\end{cases}
$$

• If $b = 0, c \neq 0$:

$$
\begin{cases}
u_1 = \sqrt{c}t, & u_2 = -u_1 \\[2mm]
v_1 = 0, & v_2 = 0
\end{cases}
$$

• If $b = c = 0$: we have the double line $u = v = 0$.

Let us note in passing that π is not flat since it has multiplicity 3 at the origin and 2 at generic points.

Since the fundamental Newton functions u^2, v^2, and uv have respective S-relative traces

$$\mathcal{T}^0_f(u^2) = 2ct^2, \ \mathcal{T}^0_f(v^2) = 2b^2, \ \text{ and } \mathcal{T}^0_f(uv) = 2at,$$

which are manifestly holomorphic on S, we have an analytic family of branched coverings of degree 2 and codimension 2 "labeled" by the morphism $F : S \times D \to \mathrm{Sym}_0^2(\mathbb{C}^2)$, sending each pair (s, t) to the symmetric functions of the local branches $f_1 = (u_1, v_1)$ and $f_2 = -f_1$. It follows that π defines a continuous family of 1-cycles of $\mathbb{C}^3$ (we will say more briefly that π is continuously geometrically flat).

The analyticity of the family is tested by looking at the S-relative traces of the Newton forms. In our case, the sheaf of 1-forms of Newton is generated by the forms udu, vdv, udv, and vdu, whose S-relative traces are $\mathcal{T}_f^1(udu) = 2ctdt$, $\mathcal{T}_f^1(vdv) = 0$, and $\mathcal{T}_f^1(vdu - udv) = 2adt$, which are clearly S-holomorphic. Thus π defines a local analytic family of 1-cycles of $\mathbb{C}^3$ (we will say that π is analytically geometrically flat).

The fibers of this morphism are evidently of type $\mathfrak{W}\mathfrak{R}^1$ (the reduction of the double line is smooth). It is easy to see that X is also of this type. Indeed, to convince oneself, one can use the base change given by the normalization (and parametrization) of S, $\varphi : \mathbb{C}^2 \to \mathbb{C}^3$ which sends (x, y) to (xy, x, y^2) to obtain the commutative diagram

$$
\begin{array}{ccc}
X_\varphi & \xrightarrow{\;\theta\;} & X \\
{\scriptstyle\tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\
\mathbb{C}^2 \times \mathbb{D} & \xrightarrow[\varphi \times Id]{} & S \times \mathbb{D}
\end{array}
$$

in which the morphism θ is a finite modification and

$$X_\varphi := \{(u, v, t, x, y) \in \mathbb{C}^2 \times \mathbb{D} \times \mathbb{C}^2 : u^2 - y^2 t^2 = 0, v^2 - x^2 = 0, uv - xyt = 0\}$$

which is three-dimensional, non-normal (non-Cohen-Macaulay), with reduced singular locus given by $\{(u, v, x, y, t) \in \mathbb{C}^5 : u = v = x = yt = 0\}$ and having two smooth irreducible components given by

$$X_\varphi^1 := \{(u, v, t, x, y) \in \mathbb{C}^2 \times \mathbb{D} \times \mathbb{C}^2 : u - yt = 0, v - x = 0\}$$

$$X_\varphi^2 := \{(u, v, t, x, y) \in \mathbb{C}^2 \times \mathbb{D} \times \mathbb{C}^2 : u + yt = 0, v + x = 0\}$$

It follows that the reduction $\mathrm{Red}X_\varphi$ has as normalization the smooth variety $X_\varphi^1 \coprod X_\varphi^2$ and is, therefore, of type $\mathfrak{W}\mathfrak{R}^1$. We deduce that X is also of type $\mathfrak{W}\mathfrak{R}^1$.

Remark 8.1. **(i)** If G is the order-two group generated by the involution σ, note that $\mathcal{G}_F$ being naturally isomorphic to $S \times D$ which is of type $\mathfrak{W}\mathfrak{R}^1$, the quotient $X/G \simeq \mathcal{G}_F$ has this type since the quotient map is a finite, surjective (and moreover open) morphism but we can not deduce directly that X is of the same type.

(ii) Note that the identification $\mathrm{Sym}_0^2(\mathbb{C}^2) \simeq \{(x, y, z) \in \mathbb{C}^3 / xy = z^2\}$ allows us to see that the form $udv - vdu$ corresponds exactly (via direct image) to the meromorphic (non-holomorphic) form $z(\frac{dx}{x} - \frac{dy}{y})$ on the cone; it naturally defines a section of the sheaf $\mathcal{L}^1_{\mathrm{Sym}_0^2(\mathbb{C}^2)}$.

(2) Analytic Family of Cusps: With the same notations, consider the ideal in $\mathbb{C}^6$ given by

$$\mathcal{I} := (u^2 - ct^5, v^2 - b^2 t^3, uv - at^4, a^2 - cb^2)$$

defining a certain subspace

$$X = \{(u, v, a, b, c, t) \in \mathbb{C}^5 \times S \times \mathbb{D} : u^2 = ct^3, v^2 = b^2 t^5, uv = at^4, a^2 - cb^2 = 0\},$$

which is irreducible, three-dimensional, non-normal, and whose two-dimensional singular locus has reduced equations $\{(u, v, a, b, c, t) \in \mathbb{C}^5 : v = ta = tb = ub = ua = 0\}$. Similarly to the previous example, the morphism $f : X \to S \times \mathbb{D}$ induced by the natural projection is finite, surjective, open, and defines an analytic family of points because the traces of the Newton forms $d(u^2)$, $d(v^2)$, $d(uv)$, and $udv - vdu$ are S-holomorphic (σ_2-invariant). Indeed, the first three obviously are; as for the last one, it suffices

to observe that the relations $u^2 = ct^3$ and $v^2 = b^2t^5$ easily give us $2udu = 3ct^2dt$, $2vdv = 5t^4b^2dt$ and, consequently,

$$udv - vdu = (5u^2b^2t^4 - 3v^2ct^2)\frac{dt}{2uv} = at^3dt$$

The morphism $\pi : X \to S$ describes an analytic family of 1-cycles which can be described as follows:

• If $b = a = 0$, $c \neq 0$: we obtain a non-reduced subspace ideal associated with $\{(u, v, t, a, b, c) : a = b = 0, u^2 = ct^3, v^2 = uv = 0\}$ whose associated cycle is generically given by

$$\left\{(\sqrt{c}t^{3/2}, 0, t, 0, 0, c)\right\} + \left\{(-\sqrt{c}t^{3/2}, 0, t, 0, 0, c)\right\}$$

whose support is the cusp $\{(u, v, t) : u^2 = ct^3, v = 0\}$.

• If $b = a = c = 0$: we obtain the double line $2.\{(u, v, t, 0, 0, 0) : u = v = 0\}$.

• If $b \neq 0$: we have the cycle

$$\left\{(\frac{a}{b}t^{3/2}, bt^{5/2}, t, 0, 0, \frac{a}{b})\right\} + \left\{(-\frac{a}{b}t^{3/2}, -bt^{5/2}, t, 0, 0, -\frac{a}{b})\right\}$$

which support is a cuspidal curve.

• If $b, c, a \neq 0$: the subspace $\{(u, v, t, a, b, c) : u^2 = ct^3, v^2 = b^2t^5, uv = at^4, cb^2 = a^2\}$ is analytically equivalent in nature to

$$\{(u, v, t) \in \mathbb{C}^3 : u^2 = t^3, v^2 = t^5, uv = t^4\}$$

describing an irreducible curve with an isolated cuspidal singularity and thus with singularities of type $\mathfrak{WR}^1$.

To show that X is also of type $\mathfrak{WR}^1$, we proceed as in the previous example. With the same notations, we obtain

$$X_\varphi{}^{14} := \{(u, v, t, x, y) \in \mathbb{C}^2 \times \mathbb{D} \times \mathbb{C}^2 : u^2 - y^2t^3 = 0, v^2 - x^2t^5 = 0, uv - xyt^4 = 0\}$$

which is again an irreducible three-dimensional non-normal subspace with a two-dimensional singular locus defined by the reduced equations $\{(u, v, t, x, y) \in \mathbb{C}^5 : u = v = ty = tx = 0\}$.

Let

$$X'_\varphi := \{(u, v, t, x, y, \alpha, \beta, \gamma) : xu^2 - \alpha.v = \alpha^2 - ty^4 = u - ty\beta = \beta^2 - t = \gamma.xt^2 - v = \gamma^2 - t = uv - xyt^4 = 0\}$$

which is a three-dimensional non-normal space possessing five irreducible components

$$X'_{\varphi,1} := \{(u, v, t, x, y, \alpha, \beta, \gamma) : u - ty\beta = v - xy.t^2 = \gamma^2 - t = \beta + \gamma = \alpha = y = 0\}$$

$$X'_{\varphi,2} := \{(u, v, t, x, y, \alpha, \beta, \gamma) : u - ty\beta = v - xy.t^2 = \gamma^2 - t = \beta + \gamma = \alpha + \gamma.y^2 = x = 0\}$$

$$X'_{\varphi,3} := \{(u, v, t, x, y, \alpha, \beta, \gamma) : u - ty\beta = v - xy.t^2 = \gamma^2 - t = \beta + \gamma = \alpha - \gamma.y^2 = x = 0\}$$

$$X'_{\varphi,4} := \{(u, v, t, x, y, \alpha, \beta, \gamma) : u - ty\beta = v - xy.t^2 = \gamma^2 - t = \beta - \gamma = \alpha + \gamma.y^2 = x = 0\}$$

$$X'_{\varphi,5} := \{(u, v, t, x, y, \alpha, \beta, \gamma) : u - ty\beta = v - xy.t^2 = \gamma^2 - t = \beta - \gamma = \alpha - \gamma.y^2 = 0\}$$

all of which are smooth. It thus appears that $\text{Red}X'_\varphi$ has locally normal crossing singularities and that its normalization $\widehat{\text{Red}X'_\varphi}$ is identified with the disjoint union $\bigsqcup_{1 \leq j \leq 5} X'_{\varphi,j}$, and it is consequently of type

$\mathfrak{WR}^1$. It is easy to see that the equations defining X'_φ allow us to recover those of X_φ. On the other hand, the canonical projection $\mathbb{C}^8 \to \mathbb{C}^5$ sending $(u, v, t, x, y, \alpha, \beta, \gamma)$ to (u, v, t, x, y) induces a generically bijective finite morphism $f^\varphi : X'_\varphi \to X_\varphi$ for which we have $f^\varphi_* \mathcal{L}^3_{X'_\varphi} = \mathcal{L}^3_{X_\varphi}$, ensuring that

$$\text{Depth}(\mathcal{L}^3_{X_\varphi}) \geq 2$$

which means that X_φ is of type $\mathfrak{WR}^1$.

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[14]It is not difficult to "projectivize" this example and the others, here X_φ is the local chart (in $Z \neq 0$) of the complex space $\{([U, V, T, Z], (x, y)) \in \mathbb{P}_3(\mathbb{C}) \times \mathbb{C}^2 : Z^2U^2 - y^2T^3 = 0, Z^3V^2 - x^2T^5 = 0, Z^2UV - xyT^4 = 0\}$.

Since θ is a proper modification induced by base change, we deduce that X is also of type $\mathfrak{WR}^1$.

(3) Analytic Families of Surfaces:

$$X = \{(u, v, t, t', a, b, c) \in \mathbb{C}^2 \times \mathbb{U} \times S : u^2 = ct^2t'^3, v^2 = b^2t'^5, uv = att'^4, a^2 = cb^2\}$$

where $\mathbb{U}$ is a relatively compact polydisk in $\mathbb{C}^2$. This subspace is of dimension 4, irreducible and non-normal, with a singular locus of dimension 3 defined by the reduced equations $\{(u, v, t, t', a, b, c) \in X : v = t'b = t'a = ua = ub = 0\}$. We have a family of surfaces.

The natural projection $q : \mathbb{C}^7 \to \mathbb{C}^5$ sending (u, v, t, t', a, b, c) to (t, t', a, b, c) induces a morphism $f : X \to S \times \mathbb{D}$ which is finite and surjective (since we have a classifying morphism $F : S \times \mathbb{U} \to \mathrm{Sym}_0^2(\mathbb{C}^2)$ that makes it a branched covering of degree 2; just as explained in the first example). The morphism $\pi : X \to S$ (the composition of the morphism $f : X \to S \times \mathbb{U}$ and the natural projection $q : S \times \mathbb{U} \to S$) is an equidimensional (and open) morphism defining an analytic family of 2-cycles (whose supports are surfaces). We can easily be convinced of this by verifying that the S-relative trace $\mathcal{T}r_{/S}(vdu - udv)$ defines an S-holomorphic function. Again, we start from the differential relations

$$2udu = 2ctt'^3dt + 3ct^2t'^2dt', \ \ 2vdv = 5b^2t'^4dt'$$

which give us

$$vdu = cvtt'^3\frac{dt}{u} + 3cvt^2t'^2\frac{dt'}{2u}, \ \ udv = 5b^2ut'^4\frac{dt'}{2v}$$

and, consequently,

$$\mathcal{T}r_{/S}(vdu - udv) = at'^3(t'dt - tdt')$$

which is clearly an S-holomorphic 1-form.

This family can be described as follows:

• If $b = a = 0, c \neq 0$, the equations reduce to $\{(u, v, t, t', a, b, c) : u^2 = ct^2t'^3, v^2 = uv = 0\}$ which represents the cycle (outside the ramification)

$$\{(\sqrt{c}t.t'^{3/2}, 0, t, t', 0, 0, c)\} + \{(-\sqrt{c}t.t'^{3/2}, 0, t, t', 0, 0, c)\}$$

whose support is the surface with equation $\{(u, 0, t, t', 0, 0, c) : u^2 = ct^2t'^3\}$, which is of type $\mathfrak{WR}^1$ (since its normalization is smooth).

• If $b = a = c = 0$, we obtain $\{(u, v, t, t', a, b, c) : u^2 = v^2 = uv = 0\}$ which is the double plane

$$2.\{(u, v, t, t', a, b, c) : u = v = 0\}$$

• If $a, b, c \neq 0$: the obtained cycle can be written generically as

$$\{(\sqrt{c}t.t'^{3/2}, bt.t'^{5/2}, t, t', a, b, c)\} + \{(-\sqrt{c}t.t'^{3/2}, -bt.t'^{5/2}, t, t', a, b, c)\}$$

with support being the surface

$$X_{(a,b,c)} := \{(u, v, t, t') : u^2 = ct^2t'^3, v^2 = b^2t'^5, uv = att'^4\}$$

Since the parameters are non-zero, we can reduce, after modifying coordinates, to checking whether the surface $X_1 := \{(u, v, t, t') : u^2 = t^2t'^3, v^2 = t'^5, uv = tt'^4\}$ is of type $\mathfrak{WR}^1$ or not. But the subspace

$$X_1' := \{(u, v, t, t', \alpha, \beta) : u - \alpha.tt' = v - \alpha.t'^2 = t' - \alpha^2 = \beta - t\alpha = 0\}$$

is a smooth surface, and it is easy to see that its equations allow us to recover those of X_1, which induces a morphism $\varphi : X_1' \to X_1$ whose finiteness and surjectivity are easily verified. It follows that X_1 is of type $\mathfrak{WR}^1$.

Let us now show that X is of type $\mathfrak{WR}^1$. To do this, we use the base change $\varphi : \mathbb{C}^2 \to \mathbb{C}^3$ sending (x, y) to (xy, x, y^2) to obtain

$$X_\varphi := \{(u, v, t, t', x, y) \in \mathbb{C}^2 \times \mathbb{U} \times \mathbb{C}^2 : u^2 - y^2t^2t'^3 = 0, v^2 - x^2t'^5 = 0, uv - xytt'^4 = 0\},$$

which is again irreducible, non-normal of dimension 4, with a singular locus defined by the reduced equations $\Sigma := \{(u, v, t, t', x, y) \in \mathbb{C}^2 \times \mathbb{U} \times \mathbb{C}^2 : u = v = xt' = ytt' = 0\}$. We then consider the subspace

$$X'_\varphi := \{(u, v, t, t', x, y, \alpha, \beta) : xu^2 - \alpha.v = v\beta - xt'^3 = \alpha^2 - t'y^4t^4 = \beta^2 - t' = ytt'\beta - u = xt'^2\beta - v = 0\}$$

which possesses 3 smooth irreducible components given by

$$X'_{\varphi,1} := \{(u, v, t, t', x, y, \alpha, \beta) : \alpha = u = t' - \beta^2 = v - x\beta.t'^2 = 0\}$$
$$X'_{\varphi,2} := \{(u, v, t, t', x, y, \alpha, \beta) : \alpha = y = u = t' - \beta^2 = v - \beta.xt'^2 = 0\}$$
$$X'_{\varphi,3} := \{(u, v, t, t', x, y, \alpha, \beta) : \alpha = t = u = t' - \beta^2 = v - \beta.xt'^2 = 0\}$$

Again, it is easy to see that the equations of X_φ are derived from those of X'_φ and that we have a generically bijective finite morphism $f^\varphi : X'_\varphi \to X_\varphi$ induced by the canonical projection from $\mathbb{C}^8$ onto the first factors of $\mathbb{C}^6$.

As in the previous examples, we deduce that X is of type $\mathfrak{WR}^1$.

Remark 8.2. In the examples above, we deliberately chose a simple parameter space given by the Whitney umbrella, which is weakly normal. However, we can take any surface of the type $S_{i,j,k} := \{(a, b, c) \in \mathbb{C}^3 : c^i b^j = a^k\}$, which is normal for $i = j = 1$, weakly normal for $i = 1, j = k = 2$, and otherwise non-weakly normal but always weakly rational (i.e. of type $\mathfrak{WR}^1$). By the same methods, we can look at more general parameter spaces with more complicated singularities and coverings of degree 3 by considering the homogeneous part of degree 3 of $\text{Sym}(\mathbb{C}^2)$ generated by the terms u^3, v^3, w^3, uv^2, uw^2, vw^2, u^2v, u^2w and wv^2.

For testing weak normality[15] (**WN**) for hypersurfaces, we use the differential criterion from [A.A.L], p.300: if $X := \{x \in \mathbb{C}^n : f(x) = 0\}$ is a pure n-dimensional hypersurface of $\mathbb{C}^n$ with Hessian form at a point x_0 defined by

$$\text{Hess}(f)_{x_0}(u) = \sum_{1 \le i,j \le n} \frac{\partial^2 f}{\partial z_i \partial z_j}(x_0)u_i u_j$$

Then

X **WN** and $\dim\text{Sing}(X) = n - 1 \implies W_4 \cap (\text{Sing}(X) \setminus \text{Sg}(\text{Sing}(X))) = S_2(X) \cap (\text{Sing}(X) \setminus \text{Sg}(\text{Sing}(X)))$

where $S_2(X) := \{x \in \text{Sing}(X) : \text{rank}(\text{Hess}(f)_x) \le 1\}$, for an n-dimensional complex space Y, $\text{Sg}(Y) = \text{Sing}(Y) \cup (\bigcup_{0 \le j \le n} Y^{(j)}$ with $Y^{(j)} := \{y \in Y : Y \text{ has a branch of dimension } j \text{ in } x\}$ and $W_4 := \{x \in \text{Sing}(X) : \dim C_4(X, x) > n\}$ with $C_4(X, x)$ the fourth Whitney tangent cone of X at x.

We can easily see that the surfaces $\{(x, y, z) \in \mathbb{C}^3 : x^2 + zy^3 = 0\}$ and $\{(x, y, z) \in \mathbb{C}^3 : z^3 + xy^2 = 0\}$ are not weakly normal.

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